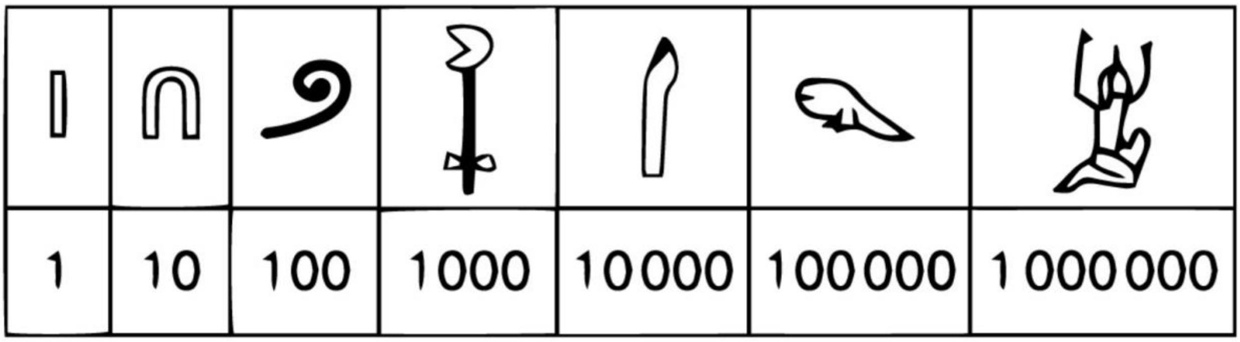
BASED ON WHOM?

COUNTING SYSTEMS

Egyptian numbers

Age of introduction: around 3000 BC.

The Egyptian number system was quite advanced at the time it was invented. Numbers were represented with the symbols shown in the figure. The position of the symbols did not have a specific value as in our system, which is called, precisely, positional. On the contrary, it was an additive system, that is, the values of the symbols were added together.



Our number 1 was easy for the Egyptians to write as they had a specific symbol: a straight vertical bar " I"

Even the number 10 was easy: an inverted U." ∩".

But now imagine how to write, "I have 8 goats." To do this, they had to repeat eight times the symbol representing the 1 and thus obtain "IIIIIIII"

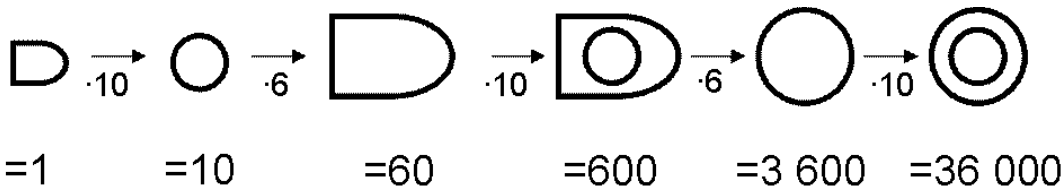
If they had 16 goats, they should have written, " ∩IIIIII" or they could have written "III∩III", which also meant 16, because, as mentioned, the position of the symbols did not matter.

It is also worth mentioning that the Egyptians invented geometry-the study of points, lines, angles, surfaces and solids. They knew how to calculate the volumes of cylinders and pyramids and the areas of different geometric shapes. The geometric concepts they developed were useful in reconstructing the boundaries of farmland along the Nile River after floods. And, certainly, the amazing Pyramids of Giza are proof that the Egyptians not only invented geometry but became masters in its use.

The counting system of the Sumerians

What they are. Counting cones are an ancient system of counting used before the invention of numbers and were used in the areas of present-day Iran and Iraq by Sumerians and Elamites around 3000 B.C. They were made of dried clay and their shape established their value.

Why? The absence of numbers posed the following problem for the Sumerians: how to represent quantities and how to do complex calculations whose resolution could not be achieved by the use of mind and hands alone? The Sumerians fabricated cones, tokens, marbles and spheres to represent different values:



Small cone / Ball / Large cone / Large cone pierced / Large sphere / Large sphere pierced

The values chosen by the Sumerians for tokens highlight the use of base 60 with auxiliary base 10 for their numerical system. Arithmetic operations were performed manually, that is, by adding or subtracting tokens: for subtractions it was often necessary first to "specify" a token of a given value into tokens of a lower value. To solve the 30 - 7 subtraction, one had to break down one of the large "tens" marbles into ten units consisting of small cones, and then remove 7 small cones, as indicated by the subtraction. This leaves 2 large marbles (tens) and 3 small ones (units), thus 23.

Fun fact: The cones were not only a calculation tool: a loan could also be taken out using a clay bulla in which the cones representing the amount in question were placed. It was then fired or dried and signed by the parties. One would break the bubble upon repayment, checking the amount. Later clay tablets were invented on which the cones–and the corresponding loan values–were drawn, thus creating numerical symbols. The tablet below shows how the quantity of different objects was marked in different boxes (3rd millennium B.C.).

The counting system of the Babylonians

Era of introduction: 1900 B.C. to 1800 B.C.

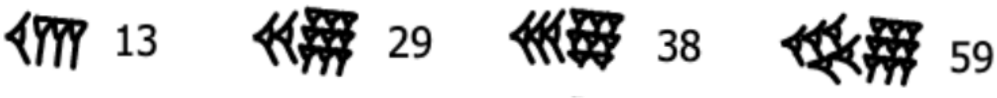
The Babylonian counting system uses the base 60 instead of the base 10 that we use today to count in the Western world. However, even today we count some things in base 60: for example, an hour has 60 minutes and a minute has 60 seconds. In a circle there are also 360 degrees $(6 \cdot 60)$; their system is not difficult to understand, partly because in the Babylonian system the position of the numbers was important i.e. it was a positional system.

The Babylonians had only two symbols:

 representing 1

 representing 10

These two symbols were used to represent the numbers up to 59. Here are

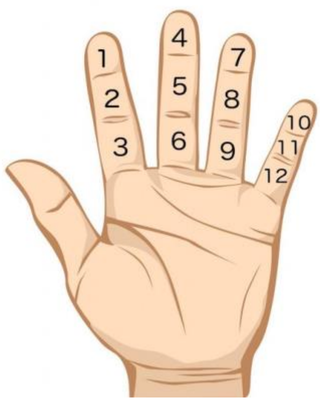


some examples

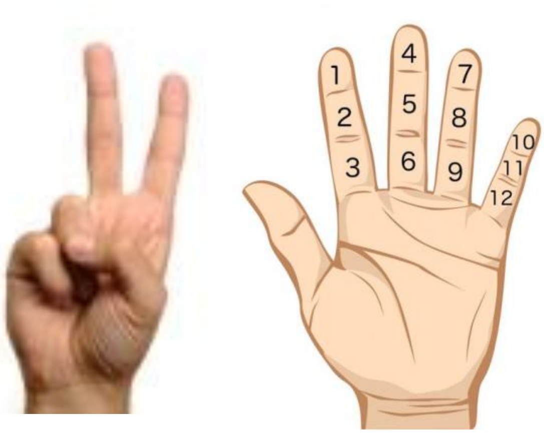
Neither did the Babylonians have a symbol for zero. However, they later "invented" a sign for zero and used it only in the middle of the number, never at either end.

The Babylonians, like us, also used their hands to count. But we have only 10 fingers. How then did they count the larger numbers?
They invented a new system.

With the thumb they counted the three segments - phalanges - of the other four fingers to arrive at 12.



They then marked the fact that they had arrived at 12 raising a finger on the other hand.
And, in a similar way, they marked the fact that they had arrived at24 by raising two fingers on one hand and pointing to the 12 on the other.
Then $2 \cdot 12 = 24$

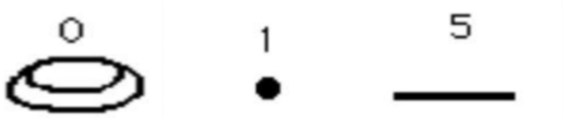


The numbers of the Maya

Time of introduction: around 500 BC.

Mayan mathematics is the most sophisticated counting system ever developed in antiquity. It uses a system based on the 20, which was probably developed using the fingers and toes to count.

The system provides only three symbols: a dot representing the value of 1, a horizontal bar representing the 5and a circle inside another circle representing the zero0. However, the Maya zero was used only as a placeholder and not for calculations.



These symbols, used in various combinations, were derived from real everyday objects: the zero from a seashell, the dot from a beanstalk, and the rod from a wooden stick.

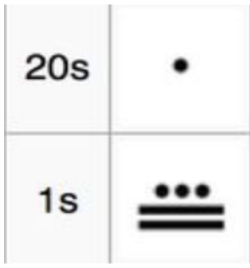
Addition and subtraction were simple, and even uneducated people could do the math needed for trade and commerce.

To add two numbers together, for example, the symbols of each number were placed side by side and then joined to make a new single number. Thus, to two bars and a single dot representing the number 11can be added a bar for five to get three bars and a dot, i.e. 16. This is true if one stays within 20, otherwise the positional value must be considered.

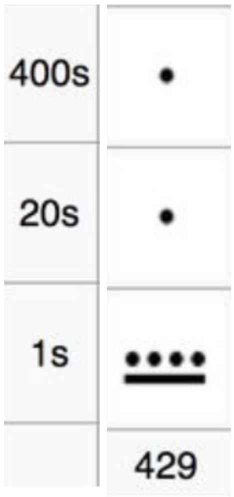
The position of one symbol relative to another was important to the Maya: the higher a symbol was placed, the greater its value. Its value increased with the powers of winds.

Here are two examples:

33 is written with a dot above the symbols of 13. The dot above represents "a twenty" or "1×20", which is added to 13.
Therefore, $(1 \times 20) + 13 = 33$.



The number 429 in Mayan symbols is represented as in the figure: the nine symbols at the bottom, a dot above representing the 20 and another dot above representing $20 \times 20 = 400$.



Roman numerals

Age of introduction: from the founding of ancient Rome 753 BC.

The Roman numerical system was used for almost 1800 years before it was replaced by the Indo-Arabic system we use today. This was in the 1300s, so only 720 years ago!
Like the Egyptians and Babylonians, the Romans had no zero. Moreover, the Roman system was additive.

Roman numerals were represented by "letters."

I = 1	C = 100
V = 5	D = 500
X = 10	M = 1000
L = 50	

These "letters" were lined up to make numbers. In Roman, 72 would be **LXXII** ie: **L** = 50, **X** = 10, **I** = 1 then $50 + 10 + 10 + 1 + 1 = 72$

For longer numbers the Romans invented a new rule. A subtractive notation was adopted, where **VIIII, for example,** was replaced by **IX** (10 - 1 = 9). This simplified the writing of long numbers somewhat, but made the calculation even more difficult.

We take the number 19. Following the first rule above, it would be written 19 in "Roman" as: **XVIIII** ($10 + 5 + 4 = 19$), but with the new rule it became XIX - i.e. $10 + 10 - 1$ equal to 19. And the number 14 **- XIIII** - becomes **XIV**.

This new rule applies only to numbers that are ten times or less the value of the previous number. For example: 1999 cannot be written as **MIM**, because **M** is a thousand times the value of **I**. In this case it should be done in this way:

1999 in Roman is **MCMXCIX**
That is, **M** (1000) + **CM** ($1000 - 100$) + **XC** ($100 - 10$) + **IX** ($10 - 1$)

The numbers of the ancient Indians

Age of introduction: started around 600 BC.

The oldest documented zero is surprisingly modern: it is found in the Chaturbhuj temple in Gwalior, Madhya Pradesh in central India, and dates from about 875 B.C. The temple is famous for being the oldest example of a written number zero: it is engraved on the temple wall, part of the number "270" clearly visible.

The Arabs brought the Indo-Arabic number system to Europe, and today it is used throughout the Western world.

Indians count like Westerners, at least up to 99999, and the position of the numbers is important. Starting from 100000, Indian number names are different.
100000 is called a hundred thousand in the West and one lakh in India.
1000000 is called one million in the West and ten lakh in India.

10000000 is called 10 million in the West and one crore in India. These are other names for even higher numbers, but we'll stop here.

Indians also write numbers differently from Westerners. The position of commas and periods varies:
100000 is written in the West as 100,000 or 100,000 (one hundred thousand) and in India 1,00,000 (one lakh);
30000000 is written in the West as 30,000,000 or 30,000,000 (thirty million) and in India 3,00,00,000 (3 crore).

How do computers count?

Introduction period: The binary system was invented around 1700, computers around 1940.

The computer knows only two numbers: 0 and 1. That's all!
0 represents "off" and 1 represents "on." "Off" and "on" refer to the electronic switch, which can be on or off. For example, turning the switch on turns the light on. In old computers, there were many switches that could be turned on and off to represent different numbers.

This system of "ones" and "zeros" is called a binary system.
The "one" and the "zero" are called bits. Bit is the abbreviated form of **BinarydigIT**.
Within a computer the bits are usually eight in number.
Eight bits are called bytes. For example, 10011010 is a byte.

Any song, movie, photo, book, image, and so on can be translated into a sequence of ones and zeros. These sequences make videos and photos visible on the computer screen. Amazing, isn't it?

The more bits/bytes a computer has, the more information (photos, text, video, music, ...) it can store.

The binary system had already been invented in 1703 by the German philosopher and mathematician Gottfried Wilhelm Leibniz. He wrote a complete documentation of the binary system that was later used by the inventors of the computer. These early machines looked very different from today's computers, but they all operated on this binary basis.

Today *smartphones* are much more powerful than early computers!

Educational insights

Proposing some historical aspects of mathematics, particularly the various number systems that have followed one another over time and were created by different cultures, and the counting tools used by them, serves several purposes:

- experience counting and number representation in ways other than what we are used to, thus opening our eyes to the multiplicity of solutions;
- understand our number system more deeply, since it is from comparing it with other systems that we can better understand how the one we use on a daily basis works;
- Compare ancient tools with those we use today;
- sense that each culture has made different choices all directed toward the same end (the desire to make counts) and that each choice is to be considered shareable, so as to raise awareness of respect for the other;
- Understand that mathematics is not a static discipline, but one that is constantly evolving;
- To perceive that mathematics was made by man for man;
- Passionate about mathematics.

With regard to elementary schooling, the different numbering systems of the various cultures that have succeeded each other over time, and the counting tools used by them, are considered among the materials of the "MaMa-mathematics for elementary school" project commissioned by the Department of Education, Culture and Sports to the Mathematics Teaching Competence Center of the Department of Formation and Learning/High Pedagogical School in Locarno, Switzerland. These materials can be downloaded free of charge at this link: https://mama.edu.ti.ch/.

In particular, it is suggested to consult:

- the *Guidelines* for having mathematical, educational and historical insights related to the various number systems of the ancients. This document also presents some historical tools, including those presented here, used by various cultures. This document may also be useful for later school levels;

- the *Context of Meaning* "Mathematics Traveling in Space and Time," a document in which insights are provided for designing meaningful learning situations related to mathematics from different cultures, used in different historical periods;
- the Teaching *Practice* "The Number Systems of the Ancients," a document in which are collected teaching proposals relating to, among others, the number systems of primitive men, the Sumerians, the Incas, the Egyptians, the Mayans, the Babylonians and the Romans the teaching practice "Different Algorithms of Calculus," where different algorithms are proposed, some of which have followed each other throughout history and have characterized different places; - the teaching practice "Figures and the Positional System," where there are many ideas related to the Indo-Arabic numeral system, including the construction of a small abacus.

- the *Teaching Sheets* designed for learners, which can be found by setting the filter "Other Number Systems" in the teaching materials search engine. In particular, we highlight: "The Speed Race," "Ten Notches," "Sumerian Numbers 1," "Sumerian Numbers 2," "Sumerian Numbers 3," "Sumerian Numbers 4," "Inca Numbers 1," "Inca Numbers 2," "Roman Numbers 1," "Roman Numbers 2," "Roman Numbers 3," "Roman Numbers 4," "Maya Numbers 1," "Maya Numbers 2, "The Mayan Numbers 3," "The Mayan Numbers 4," "The Egyptian Numbers 1," "The Egyptian Numbers 2," "The Egyptian Numbers 3," "The Egyptian Numbers 4," "Race Between Systems," "Comparing Systems," "Ancient Calculations," "The Use of the Abacus," "Let's Know the Abacus," "Decimals What a Passion," "Laura Don't Get Distracted."

In addition, there are 22 comics related to important mathematicians throughout history in the collection "Mathematicians in Comics," which can be downloaded free online or purchased in hard copy published by Daedalus Publishing House. In particular, for an in-depth look at the birth of our numbering system see the comic strip by mathematician Al-Khwārizmī (9th cent.).

Also available for active middle school teachers are teaching materials called "Mathematics in History," supplemented by student worksheets that can be downloaded from the ScuolaLab portal (where you must register to download the documents).

The following storybooks, suitable as early as elementary school, focus on counting and can be related to the use of the abacus and other counting tools:
- Abedi, I. (2002). *One, two, three... 99 sheep!* Daisy Editions.
- Bellei, M. (2020). *Cities of numbers*. Fatatrac.
- Cerasoli, A. (2012). *The great invention of Bubal*. Emme Editions.
- Cerasoli, A. (2019). *The five-fingered sisters*. Science editorial.
- Chermayeff, I. (2014). *Blind mice and other numbers*. Corraini.
- D'Angelo, S. (2008). *Never count on mice*. Topipittori.
- Fromental, J. L., & Jolivet, J. (2010). *10 little penguins*. The Beaver.
- Giusti, A. (2011). *Awa teaches counting*. The garden of Archimedes.
- Ohmura, T. (2011). *Everyone in the queue!* Babalibri.
- Tolstoy, A. (1999). *The giant turnip*. Fabbri.
- Urberuaga, E. (2015). *A black thing*. Lapis.

From this point of view, the rich collection of reviews "100 illustrated books between Italian and mathematics: a bibliography with teaching cues" written Demartini and Sbaragli is noteworthy.

The following narrative books, suitable as early as elementary school, focus on the history of mathematics:
- Cerasoli, A. (2012). *The great invention of Bubal*. Emme Editions.
- Giusti, A. (2011). *Awa teaches counting*. The garden of Archimedes.
- Petti, R. (2008). *Uri, the little Sumerian*. The garden of Archimedes.
- Petti, R. (2008). *Ahmose and the 999'999 lapis lazuli*. The garden of Archimedes.

In addition, for in-depth historical and educational study aimed at adults, the following references are recommended:
- Boyer, C.B. (1982). *History of mathematics*. Mondadori.
- Bunt, L., Jones, P. S., & Bedient, J. D. (1987). *The historical roots of elementary mathematics*. Zanichelli.
- Ifrah, G. (1989). *Universal history of numbers*. Arnoldo Mondadori. [Original ed. in French: 1981).
- D'Ambrosio, U. (2002). *Ethnomathematics*. Pythagoras.

D'Amore, B., & Sbaragli, S. (2017). *Mathematics and its history. I. From origins to the Greek miracle*. Daedalus editions.
- D'Amore B., & Sbaragli S. (2018). *Mathematics and its history: from the Greek sunset to the Middle Ages*. Daedalus editions.

- Fontana Bollini, V., & Lepori, G. (2019). A history of mathematics in middle school: the quadrature of plane figures. *Didactics Of Mathematics. From Research to Classroom Practices,* (6), 131-150.
- Nicosia, G. G. (2008). *Numbers and cultures*. Erickson.
- Vecchi, N. (2010). *How numbers came into being. An educational journey from primitive men to the abacus*. Rome: Carocci.

Also reposted below are some articles where the abacus is used as a tool. Some bring it up as an example, others compare it with other artifacts such as the pascaline.

- Bartolini-Bussi, M. G., & Mariotti, M. A. (2008). Semiotic mediation in the mathematics classroom: Artifacts and signs after a Vygotskian perspective. *Handbook of international research in mathematics education, 746*.
- Bartolini-Bussi, M. G., & Boni, M. (2003). Instruments for semiotic mediation in primary school classrooms. *For the learning of mathematics*, *23*(2), 15-22.
- Maschietto, M. (2013). Systems of instruments for place value and arithmetical operations: An exploratory study with the Pascaline. *Education*, *3*, 221-230.

AS AHMES, ASHA AND MAYA BECAME FRIENDS AND LEARNED TO "COUNT" ON EACH OTHER.

(by Antoinette Mira)

3700 years ago an Egyptian boy, Ahmes-who at age 40 became a famous scribe and transcribed the Rhind papyrus (1650 B.C.E.-and a young Indian girl, Asha, attended the same school.
Impossible, you will say. This is a fairy tale where anything is possible! Just close your eyes and let your imagination fly.

On the first day of school Asha arrived in class with a bag full of mangoes.

"If we want to become friends, we have to learn to understand each other"-Asha told Ahmes-"let's start with a common system of symbols to indicate numbers, so every day I will bring you fruits and we can count how many I have given you."

"You gave me a mango today," Ahmes said in thanks, and drew a vertical line in the sand

I

"Even the Brahmic script of ancient India in the 3rd century B.C. used the symbol I to denote the number 1!" exclaims Asha, proud of her ancestors.

"How old are you Ahmes?"
"Ask the reader!" he replied, winking ... at the reader of course.

Number 1 is thus written similarly by both Ahmes and Asha: a solid starting point for their friendship. This is not surprising. Virtually all cultures today, as well as in the past, use a similar symbol to represent the 1 just as we do. The practice is tens of thousands of years old and long predates writing, which is only five thousand years old. It seems to have been started by hunter-gatherers who, instead of writing in the sand as Ahmes had done, used bones or sticks to record quantities by carving indentations so that a trace would remain even after a rain that would erase the marks drawn on the ground. Carved bones and sticks also had the advantage of being able to be transported from place to place, and you know, hunter-gatherers were nomadic: when a land was no longer fertile or animals dwindled due to migration, they too moved from place to place in search of new food.
"Let us now turn to number 2," Ahmes continues and traces two lines on the ground with his finger to indicate that, on the second day of school, he received 2 juicy mangoes as a gift from Asha:

II

And the next day, to write number 3 Ahmes uses, of course, 3 lines:

III

Unlike their contemporaries, such as the Mesopotamians, the Egyptians did not group their I's in specific patterns. So for example, the three bars of Ahmes can be placed either horizontally or vertically.

So far, so simple. But think about the number 7: It was easy to get confused when reading it, especially from a distance:

IIIIIII

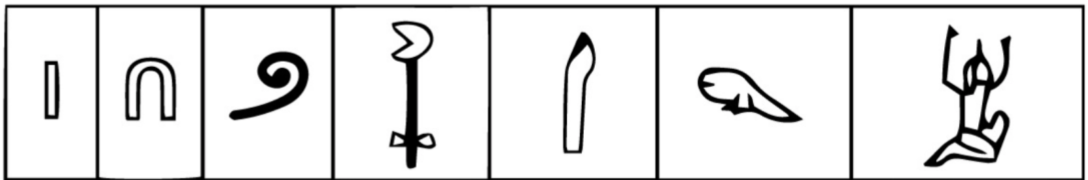
Asha then suggests some shortcuts: "If we take the two bars representing two mangoes and put them horizontally, so much for you it makes no difference, and as we trace them in the sand, to be faster in writing, we just raise our finger, that's what happens."

Thus was born the symbol representing the number 2 as we know it today.

And in a similar way we have for example the number 3 and 7 more or less as we would write it today:

While it is easy to carve a straight line with a knife on a piece of wood, carving a curved shape, like our 2, would be unnecessarily difficult. So Ahmes found Asha's idea interesting, but stuck to his symbols.

Ahmes agreed, however, that a new symbol was needed for the number 10 so that it would not be confusing to count all those sticks one in a row. So too for the number 100, 1,000, 10,000 and 100,000. A special symbol was then needed to indicate very large numbers ... and we will come back to that.



1	10	100	1000	10000	100000	1000000

Symbols used by Ahmes.

Ahmes had an interesting way of indicating addition and subtraction operations through a symbol in hieroglyphic writing. It represented a man running: toward the quantities if they were to be added, in the opposite direction if they were to be subtracted.

Ahmes also had a special symbol, which we would call "zero" today. But zero was used by Ahmes only in architecture to indicate the ground floor of a building such as a pyramid. To count the floors below ground Ahmes used what we today call "negative numbers." Now, as then, it was enough to put a particular sign in front of the number to indicate that that floor was below ground. Second floor below ground: -1 we would write.

Asha found the idea of the "zero" symbol very interesting and decided to make even wider use of it, not only in architecture but in general to represent numbers in a totally different way, the positional number system.

This idea was secretly passed from Asha to Brahmagupta a mathematician and astronomer who in the 6th century used a small black dot to denote this new symbol. Thus was born what we call zero today.

To indicate the number one million Ahmes depicted a man with his arms raised toward the sky. In general this symbol was used to represent extremely large numbers as we would do today if we wanted to indicate with our arms that we love someone very much: we would spread them wide to symbolically contain all our love!

Maya, who attends the same school and comes from Guatemala-in Central America, the cradle of Maya civilization as early as 750 B.C.-wants to

have his say on systems for representing numbers, on the strength of the fact that the Maya had devised a sophisticated counting system to take into account, for example, the time between two religious rituals, between particular astronomical events such as eclipses, between periods of greater or lesser fertility of the earth, or again, to avoid embarking on journeys on unlucky days and instead arrive at their destination on auspicious days.

"We have 10 fingers and 10 toes, so why not count in base 20?" asked Maya. It, actually, seems very natural, considering also that in those ancient civilizations people mostly walked barefoot and therefore toes... were within reach! "Explain your idea better", asked Ahmes and Asha.

"Let's imagine that we only have three symbols," Maya continued:
. points, FRIJOL (bean), for the unit
_ lines, PALITO (stick or toothpick) for #5
0 and a shell, the symbol for zero

"In my opinion, only three symbols are too few to write all the numbers!" argues Ahmes who was used to using many more.

But here is Maya's brilliant idea.
The symbol changes value depending on its position. An idea the Egyptians had not thought of, and, in fact, for Ahmes the order in which the symbols were written was irrelevant.

Imagine, Maya said, building a temple of numbers—and you know, the Maya were masters of building magnificent temples.

As one ascends to the higher floors in a temple one gains value, and indeed, at the top of temples we usually find precious altars for sacrifices to deities.
Now we think that the symbols on the higher floors-written above-are worth more than those on the lower floors. Each time you go up a floor the symbols are worth 20 (like fingers and toes combined) times more.

For example, a bean on the ground floor is worth 1:
. one

A bean on the second floor is worth 20 times as much, so it is worth 20. Here then is the number 21:
. twenty.
. one

And the number 22
. twenty.
.. two.

And again number 41:
.. twenty x 2 = forty
. one

And, if we go up another floor here is number 421

. 1 x 20 x 20 = 400 = four hundred
. twenty.
. one

The same applies to the symbol "little wood": if it is placed on the ground floor we have the number 5:

_ five

But if the little wood goes up one floor here is where its value goes from 5 to 5x20 or 100. And so Maya writes the number 105:
_ 5 x 20 = 100
_ 5

And the number 101 becomes:
_ hundred
. one

And here is the surprising and magical usefulness of the zero: to write the number 20 you need a special symbol, the zero in fact!

The number 20 is then written by Maya as follows:
. twenty.
0

Number 400:
. four hundred
0
0

And if we add a bean on the ground floor we reach 401:
. four hundred
0
. one

If we put a stick on the ground floor instead, we have the number 405:
. four hundred
0
_ five

Easy exclaimed Asha! Who decided to keep Maya's "positional" idea (symbols have a different value depending on position), but wanted to change the base from 20 to 10 (thus using only the fingers of the hands as a reference) and instead of just 3 symbols decided to work with 10 symbols.

Instead of ascending to higher planes, Asha symbols change value by moving to the left, and each time a symbol moves left one position its value is multiplied by 10:

1
10
100

And the same for symbol 5:

5
50
500

"Easy isn't it?" cried Asha happy with her new and elegant way of representing numbers.

And so it was that Ahmes, Asha and Maya became best friends, lived happily ever after being able to rely on each other...for eternity, that is, for an infinite time.
And about infinity we will talk in another story!