DIAMO I NUMERI!

DOSSIER EDUCATIONAL

INTRODUCTION

The mathematician is an imaginative person. Like the poet or the painter. I always tell children the story of Gauss, who lived in Germany between the 18th and 19th centuries. One day at school the teacher, as punishment, gives the task of adding up all the numbers from one to 100. Crazy stuff! But Gauss comes up with a trick: he adds up 1+100, that is, the first and the last number in the list; then 2+99, 3+98... And he discovers that it always makes 101. Since there are 100 numbers, he has to sum 50 pairs that together make 101. Gauss multiplies 101×50 and turns in the assignment first. He was 8 years old and a very imaginative child-he became one of the most important mathematicians in history."

Bruno D'Amore, mathematician and pedagogue
University of Bologna

Nmb3d by numb3rs! (DIN) is a project that aims to bring mathematics closer to young people and adult audiences without taking the place of school, but creating a nonformal way of encountering knowledge where the individual's experience, pleasure and emotions can come into play. Mathematics, from being an excellent science, a synonym for wisdom, a bridge between cultures, has too often become a hated or resented discipline. Paradoxically, this is happening at a time when rains of data, of numbers that are increasingly difficult to handle and interpret, are being generated at every moment. From an early age we must familiarize ourselves with all this, little by little, to become conscious citizens of a world in which knowing how to count also helps us live better.

Who we are

The project Numb3d by numb3rs! (Diamo i numeri!) - funded by the Swiss National Science Foundation (SNSF Agora)-came about as a result of close collaboration between Prof. Antonietta Mira, the scientific head of the project (Professor of Statistics at the University of Italian Switzerland and the University of Insubria), and L'ideatorio, which oversaw the set-up and organization with the advice of the Mathematical Society of Ticino (SMASI) and the support of several other local entities. This documentation was written by several hands and in several stages. The first draft was written by the staff of L'ideatorio in collaboration with Prof. Antonietta Mira and Federica Bianchi of the Faculty of Economics at USI. The dossier was then enriched thanks to the collaboration of Prof. Paola Mira (mathematics and science teacher at the middle school in Casorate Primo, Italy), Prof. Silvia Sbaragli (SUPSI) and Luca Crivelli (SUPSI) for educational insights and Dr. Fabio Meliciani.

It is not meant to be a science textbook but a cue for teachers visiting the exhibition and corresponds to the information in the Numb3d by numb3rs!

Organization of the Dossier

The dossier consists of three parts in line with the organization of the DIN exhibition: Fingers (mathematics), DICE, (probability) and DATA (statistics, big data, data science).

Disclaimer

This document was only distributed to the animators of the exhibition Numb3d by numb3rs! and to teachers who visited or animated the exhibition, with a request not to disseminate the document to others. All images are Creative Common License.

Video

In these 3 videos you can view the locations of the 3 sections of the exhibition in the setup in Pavia.
It may be that the version of the exhibition you visit does not have all the stations.
Detailed instructions on how to build individual stations are available at this link:

https://fabiomelicianiscience.com/

DIGITS (mathematics)
https://www.youtube.com/watch?v=mGw8cU7xoeA&t=4s

DICE (probability)
https://www.youtube.com/watch?v=7xIifSagq_E

DATA (statistics, data science)
https://youtu.be/HFgvPfdKOaU

At these links you can listen to two facilitators of the exhibition in Pavia explaining some of the stations:

Ilaria Lago explains the horse race
https://youtu.be/zMVAjcsOFJU

Samuel explains the colorful SUDOKU.
https://youtu.be/FZWAXNJZJaI

NUMB3D BY NUMB3RS!

DIGITS

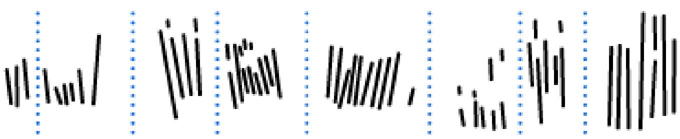
MADE TO COUNT

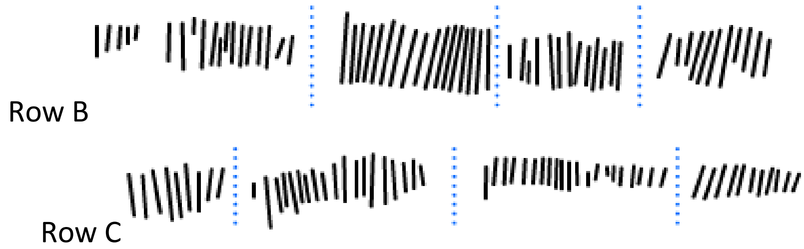
THE ISHANGO BONE

What it is. The earliest evidence of human use of numbers dates back more than 35,000 years and consists of bones carved with notches that are thought to indicate some kind of counting-days, animals? The most famous ancient find is the bone from Ishango, a village located in the Democratic Republic of the Congo, dated between 20,000 BC and 18,000 BC and dating to the Upper Paleolithic.

Why? The organization of the notches into three asymmetrical pairs implies that this arrangement was intentionally intended. What can be inferred is a first step toward the construction of a true number system. Line A-reading from left to right-begins with 3 notches and is followed by 6 notches, twice as many. The same for the following: 4 notches, then 8, and then reversing the system with 10 being followed by 5. These numbers, then, suggest an understanding of multiplication and division by 2. In addition, numbers of notches on either side of rows B and C seem to indicate a more important calculating skill. The numbers are all odd: 9, 11, 13, 17, 19, and 21. Those notched in row B are all prime numbers between 10 and 20, while those on row C are made up following the rule 10 + 1, 10 - 1, 20 + 1, and 20 - 1.

Row A

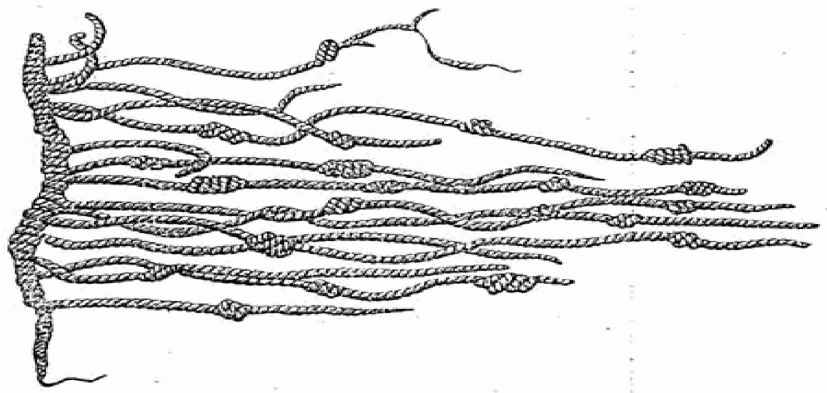




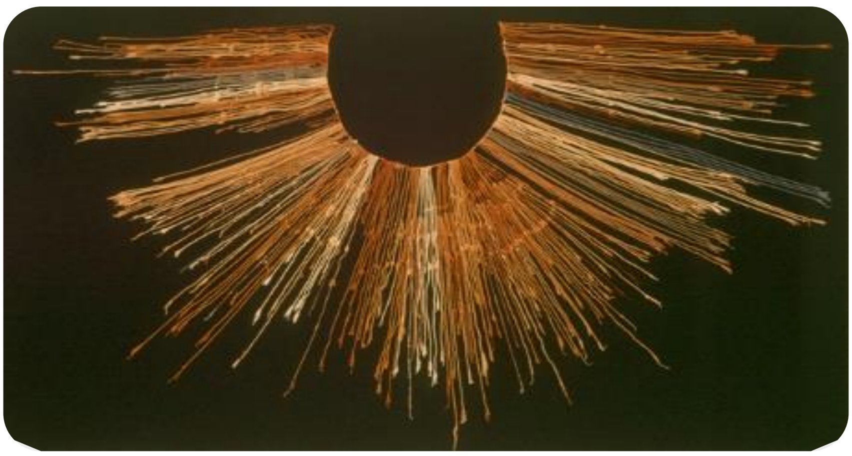
Fun fact: The Ishango bone is the fibula of a baboon with a sharp quartz flake grafted to one end, probably used for carving. Among the hypotheses that best explain the arrangement of these notches, scholars assess it to be the computation of a day sequence and specifically a lunar month. The bone was discovered in 1960 by Jean de Heinzelin de Braucourt (Belgium) during an exploration of the former Belgian Congo. It was found near Ishango, on the Uganda-Congo border. The people who lived there in 20,000 B.C. may have been among the first to use a numerical system. The find is on display at the Royal Museum of Natural Sciences in Brussels.

THE QUIPU

What it is. In the Quechua language (an indigenous language native to South America) the quipur quipu is a set of ropes knotted at regular intervals and tied to a larger, shorter rope that holds them up.



Why? Some civilizations that developed in Latin America, which had not yet invented numbers, had to engineer themselves to solve mathematical problems concerning census, taxes, counting items bought or sold, astronomical calculations, description of historical and economic events, and of course arithmetic operations. It is possible that the quipu was used by Inca administrators to perform addition, subtraction, multiplication and division for the inhabitants. Some nodes recorded on the quipu were not numbers, but "labels" of numbers that were used as a code, just as we use numbers to indicate objects, places, people, etc. Other elements of the quipu, such as the position and distance between strings, and the colors used, also represented information.



Fun fact: To this day, the quipu remains an object whose use has not been totally explained. Quipu were made to remain unaltered: after being soaked and allowed to dry, they were glued with special resins. Even today, a simpler version of the quipu is used by Peruvian and Bolivian herders. Quipus may consist of only a few strings, but some go as high as 2,000.

THE COUNTING CONES

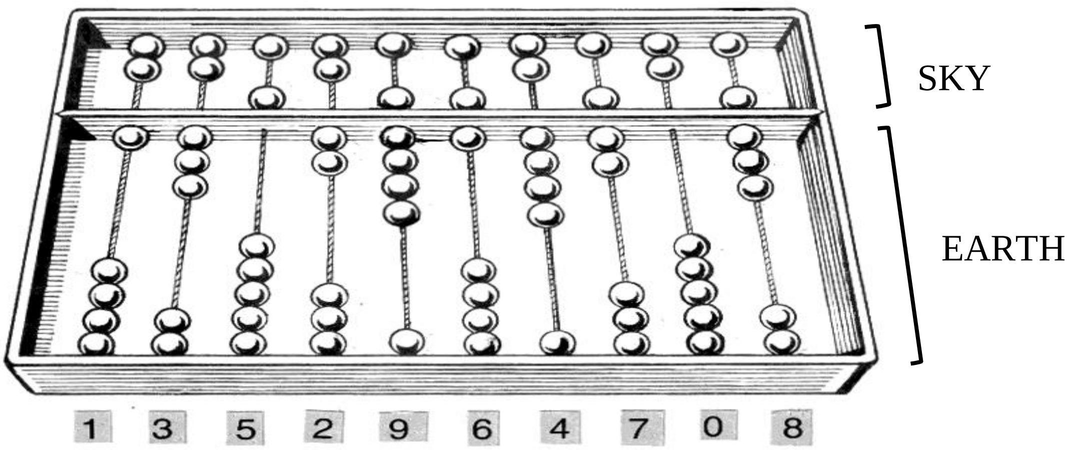
In the display case found in the exhibit, along with the Ishango bone and the quipu appear the cones used by the Sumerians for counting. We have postponed the discussion of counting cones in the next chapter (Based on whom?), under the heading "The Sumerian Counting System."

THE ABACUS

What it is. The word abacus comes from the Greek word *abaks* meaning table, which in turn comes from the Semitic word *abaq* meaning sand. Used as early as the time of the Greeks and Romans, the earliest abacus consisted of a table covered with a thin layer of sand where calculations were made with a stylus.

The Chinese-type abacus consists of a series of rods that, from right to left, indicate the different orders of value: on the first rod the units, on the second the tens, on the third the hundreds, on the fourth the thousands, etc. Each rod contains seven balls and a bar that divides five of them-in the part called earth-from two others-in the part called sky. The value of each ball in "sky" corresponds to five times those in "earth."

There is also a six-ball model (Japanese abacus), which is more complicated in use, with 5 balls for "earth" and 1 for "sky."



Why. The abacus is an ancient tool invented by man for solving calculations and is still used in Chinese schools for teaching today. To represent a number you move the balls toward the horizontal rod, remembering that the balls at the top are worth five times those at the bottom of the same rod. In the figure above the number is 1'352'964'708.

Fun fact: In ancient China, abacuses with bamboo sticks were used, later tables prevailed on which division lines and columns were marked to indicate the different orders of units in the number system in use. Tokens representing the numbers were then placed on these lines so that any kind of calculation could be carried out.

An important shortcoming of the abacus is that it does not allow past operations to be fixed, so if a mistake is made, everything has to be repeated from the beginning. Japanese companies got around this problem by having three abacists perform the same calculation at the same time: if the three solutions were identical, then the solution was correct.

To learn how to represent number and do operations with the abacus we recommend these videos

https://www.youtube.com/watch?v=FTVXUG_PngE
https://www.youtube.com/watch?v=I8EZvig5fOU
https://www.youtube.com/watch?v=YQLjDD9SzGA COMPLEMENTARY METHOD

https://www.youtube.com/watch?v=22NdwzuEZi4 EXCHANGE METHOD
https://www.youtube.com/watch?v=r0aKV3HqDzA COMPLEMENTARY METHOD

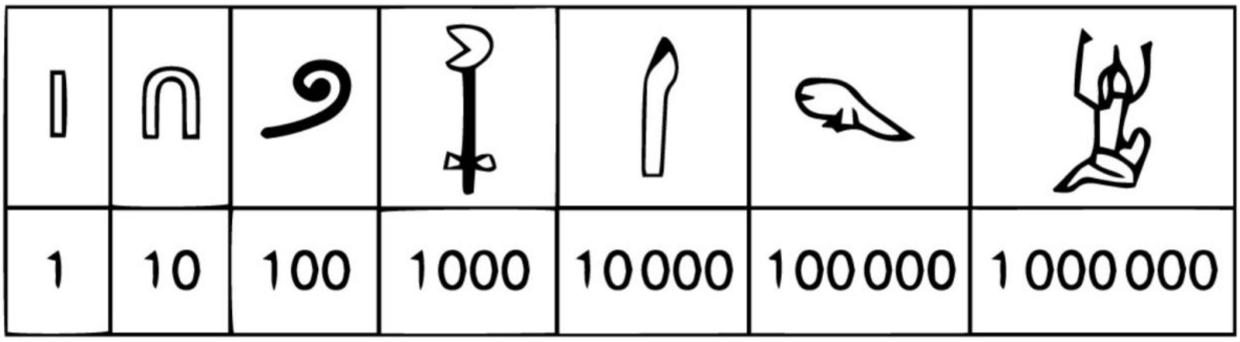
BASED ON WHOM?

COUNTING SYSTEMS

Egyptian numbers

Age of introduction: around 3000 BC.

The Egyptian number system was quite advanced at the time it was invented. Numbers were represented with the symbols shown in the figure. The position of the symbols did not have a specific value as in our system, which is called, precisely, positional. On the contrary, it was an additive system, that is, the values of the symbols were added together.



Our number 1 was easy for the Egyptians to write as they had a specific symbol: a straight vertical bar " **I**"

Even the number 10 was easy: an inverted U." ∩".

But now imagine how to write, "I have 8 goats." To do this, they had to repeat eight times the symbol representing the 1 and thus obtain "**IIIIIIII**"

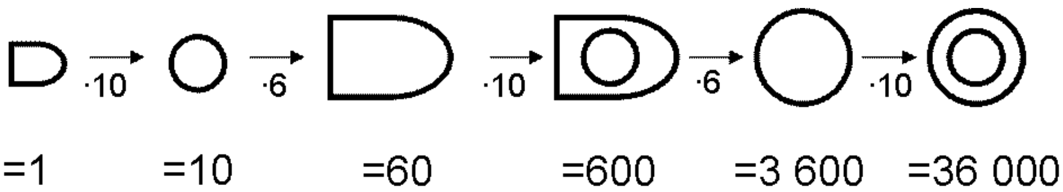
If they had 16 goats, they should have written, " ∩**IIIIII**" or they could have written "**III**∩**III**" , which also meant 16, because, as mentioned, the position of the symbols did not matter.

It is also worth mentioning that the Egyptians invented geometry-the study of points, lines, angles, surfaces and solids. They knew how to calculate the volumes of cylinders and pyramids and the areas of different geometric shapes. The geometric concepts they developed were useful in reconstructing the boundaries of farmland along the Nile River after floods. And certainly the amazing Pyramids of Giza are proof that the Egyptians not only invented geometry but became masters in its use.

The counting system of the Sumerians

What they are. Counting cones are an ancient system of counting used before the invention of numbers and were used in the areas of present-day Iran and Iraq by Sumerians and Elamites around 3000 B.C. They were made of dried clay and their shape established their value.

Why? The absence of numbers posed the following problem for the Sumerians: how to represent quantities and how to do complex calculations whose resolution could not be achieved by the use of mind and hands alone? The Sumerians fabricated cones, tokens, marbles and spheres to represent different values:



Small cone / Ball / Large cone / Large cone pierced / Large sphere / Large sphere pierced

The values chosen by the Sumerians for tokens highlight the use of base 60 with auxiliary base 10 for their numerical system. Arithmetic operations were performed manually, that is, by adding or subtracting tokens: for subtractions it was often necessary first to "specify" a token of a given value into tokens of a lower value. To solve the 30 - 7 subtraction, one had to break down one of the large "tens" marbles into ten units consisting of small cones, and then remove 7 small cones, as indicated by the subtraction. This leaves 2 large marbles (tens) and 3 small ones (units), thus 23.

Fun fact: The cones were not only a calculation tool: a loan could also be taken out using a clay bulla in which the cones representing the amount in question were placed. It was then fired or dried and signed by the parties. One would break the bubble upon repayment, checking the amount. Later clay tablets were invented on which the cones-and the corresponding loan values-were drawn, thus creating numerical symbols. The tablet below shows how the quantity of different objects was marked in different boxes (3rd millennium B.C.).

The counting system of the Babylonians

Era of introduction: 1900 B.C. to 1800 B.C.

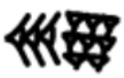
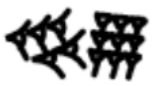
The Babylonian counting system uses the base 60 instead of the base 10 that we use today to count in the Western world. However, even today we count some things in base 60: for example, an hour has 60 minutes and a minute has 60 seconds. In a circle there are also 360 degrees ($6 \cdot 60$).Their system is not difficult to understand, partly because in the Babylonian system the position of the numbers was important i.e. it was a positional system.

The Babylonians had only two symbols:

 representing 1

 representing 10

These two symbols were used to represent the numbers up to 59. Here are some examples:

 13  29  38  59

Neither did the Babylonians have a symbol for zero. However, they later "invented" a sign for zero and used it only in the middle of the number, never at either end.

The Babylonians, like us, also used their hands to count. But we have only 10 fingers. How then did they count the larger numbers?
They invented a new system.

With the thumb they counted the three segments - phalanges - of the other four fingers to arrive at 12.

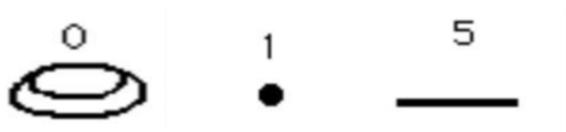
They then marked the fact that they had arrived at 12 raising a finger on the other hand.
And, in a similar way, they marked the fact that they had arrived at24 by raising two fingers on one hand and pointing to the 12 on the other. Then $2 \cdot 12 = 24$

The numbers of the Maya

Time of introduction: around 500 BC.

Mayan mathematics is the most sophisticated counting system ever developed in antiquity. It uses a system based on the 20, which was probably developed using the fingers and toes to count.

The system provides only three symbols: a dot representing the value of 1, a horizontal bar representing the $5$and a circle inside another circle representing the zero0. However, the Maya zero was used only as a placeholder and not for calculations.



These symbols, used in various combinations, were derived from real everyday objects: the zero from a seashell, the dot from a beanstalk, and the rod from a wooden stick.

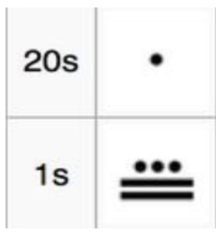
Addition and subtraction were simple, and even uneducated people could do the math needed for trade and commerce.

To add two numbers together, for example, the symbols of each number were placed side by side and then joined to make a new single number. Thus, to two bars and a single dot representing the number 11can be added a bar for five to get three bars and a dot, i.e. 16. This is true if one stays within 20, otherwise the positional value must be considered.

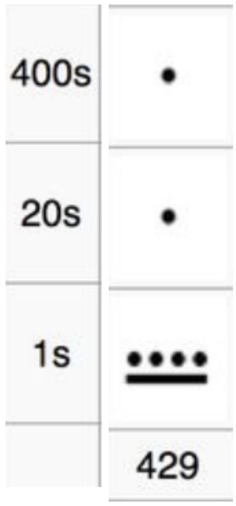
The position of one symbol relative to another was important to the Maya: the higher a symbol was placed, the greater its value. Its value increased with the powers of winds.

Here are two examples:

33 is written with a dot above the symbols of 13. The dot above represents "a twenty" or "1×20", which is added to 13.
Therefore, $(1 \times 20) + 13 = 33$.



The number 429 in Mayan symbols is represented as in the figure: the nine symbol at the bottom, a dot above representing the 20 and another dot above representing $20 \times 20 = 400$.



Roman numerals

Age of introduction: from the founding of ancient Rome 753 BC.

The Roman numerical system was used for almost 1800 years before it was replaced by the Indo-Arabic system we use today. This was in the 1300s, so only 720 years ago!
Like the Egyptians and Babylonians, the Romans had no zero. Moreover, the Roman system was additive.

Roman numerals were represented by "letters."

I = 1
V = 5
X = 10
L = 50
C = 100
D = 500
M = 1000

These "letters" were lined up to make numbers. In Roman, 72 would be **LXXII** ie: **L** = 50, **X** = 10, **I** = 1 then $50 + 10 + 10 + 1 + 1 = 72$

For longer numbers the Romans invented a new rule. A subtractive notation was adopted, where **VIIII, for example,** was replaced by **IX** ($10 - 1 = 9$). This simplified the writing of long numbers somewhat, but made the calculation even more difficult.

We take the number 19. Following the first rule above, it would be written 19 in "Roman" as: **XVIIII** ($10 + 5 + 4 = 19$), but with the new rule it became XIX - i.e. $10 + 10 - 1$ equal to 19. And the number 14 **- XIIII** - becomes **XIV**.

This new rule applies only to numbers that are ten times or less the value of the previous number. For example: 1999 cannot be written as **MIM**, because **M** is a thousand times the value of **I**. In this case it should be done in this way:

1999 in Roman is **MCMXCIX**
That is, **M** (1000) + **CM** ($1000 - 100$) + **XC** ($100 - 10$) + **IX** ($10 - 1$)

The numbers of the ancient Indians

Age of introduction: started around 600 BC.

The oldest documented zero is surprisingly modern: it is found in the Chaturbhuj temple in Gwalior, Madhya Pradesh in central India, and dates from about 875 B.C. The temple is famous for being the oldest example of a written number zero: it is engraved on the temple wall, part of the number "270" clearly visible.

The Arabs brought the Indo-Arabic number system to Europe, and today it is used throughout the Western world.

Indians count like Westerners, at least up to 99999, and the position of the numbers is important.
Starting from 100000, Indian number names are different.
100000 is called a hundred thousand in the West and one lakh in India.
1000000 is called one million in the West and ten lakh in India.
10000000 is called 10 million in the West and one crore in India.
These are other names for even higher numbers, but we'll stop here.

Indians also write numbers differently from Westerners. The position of commas and periods varies:
100000 is written in the West as 100,000 or 100,000 (one hundred thousand) and in India 1,00,000 (one lakh);
30000000 is written in the West as 30,000,000 or 30,000,000 (thirty million) and in India 3,00,00,000 (3 crore).

How do computers count?

Introduction period:
The binary system was invented around 1700, computers around 1940.

The computer knows only two numbers: 0 and 1. That's all!
0 represents "off" and 1 represents "on." "Off" and "on" refer to the electronic switch, which can be on or off. For example, turning the switch on turns the light on. In old computers, there were many switches that could be turned on and off to represent different numbers.

This system of "ones" and "zeros" is called a binary system.
The "one" and the "zero" are called bits. Bit is the abbreviated form of **BinarydigIT**.
Within a computer the bits are usually eight in number.
Eight bits are called bytes. For example, 10011010 is a byte.

Any song, movie, photo, book, image, and so on can be translated into a sequence of ones and zeros. These sequences make videos and photos visible on the computer screen. Amazing, isn't it?

The more bits/bytes a computer has, the more information (photos, text, video, music, ...) it can store.

The binary system had already been invented in 1703 by the German philosopher and mathematician Gottfried Wilhelm Leibniz. He wrote a complete documentation of the binary system that was later used by the inventors of the computer. These early machines looked very different from today's computers, but they all operated on this binary basis.

Today *smartphones* are much more powerful than early computers!

Educational insights

Proposing some historical aspects of mathematics, particularly the various number systems that have followed one another over time and were created by different cultures, and the counting tools used by them, serves several purposes:

- experience counting and number representation in ways other than what we are used to, thus opening our eyes to the multiplicity of solutions;
- understand our number system more deeply, since it is from comparing it with other systems that we can better understand how the one we use on a daily basis works;
- Compare ancient tools with those we use today;
- sense that each culture has made different choices all directed toward the same end (the desire to make counts) and that each choice is to be considered shareable, so as to raise awareness of respect for the other;
- Understand that mathematics is not a static discipline, but one that is constantly evolving;
- To perceive that mathematics was made by man for man;
- Passionate about mathematics.

With regard to elementary schooling, the different numbering systems of the various cultures that have succeeded each other over time, and the counting tools used by them, are considered among the materials of the "MaMa-mathematics for elementary school" project commissioned by the Department of Education, Culture and Sports to the Mathematics Teaching Competence Center of the Department of Formation and Learning/High Pedagogical School in Locarno, Switzerland. These materials can be downloaded free of charge at this link: https://mama.edu.ti.ch/.

In particular, it is suggested to consult:

- the *Guidelines* for having mathematical, educational and historical insights related to the various number systems of the ancients. This document also presents some historical tools, including those presented here, used by various cultures. This document may also be useful for later school levels;
- the *Context of Meaning* "Mathematics Traveling in Space and Time," a document in which insights are provided for designing meaningful learning situations related to mathematics from different cultures, used in different historical periods;
- the Teaching *Practice* "The Number Systems of the Ancients," a document in which are collected teaching proposals relating to, among others, the number systems of primitive men, the Sumerians, the Incas, the Egyptians, the Mayans, the Babylonians and the Romans the teaching practice "Different Algorithms of Calculus," where different algorithms are proposed, some of which have followed each other throughout history and have characterized different places; - the teaching practice "Figures and the Positional System," where there are many ideas related to the Indo-Arabic numeral system, including the construction of a small abacus.

- the *Teaching Sheets* designed for learners, which can be found by setting the filter "Other Number Systems" in the teaching materials search engine. In particular, we highlight: "The Speed Race," "Ten Notches," "Sumerian Numbers 1," "Sumerian Numbers 2," "Sumerian Numbers 3," "Sumerian Numbers 4," "Inca Numbers 1," "Inca Numbers 2," "Roman Numbers 1," "Roman Numbers 2," "Roman Numbers 3," "Roman Numbers 4," "Maya Numbers 1," "Maya Numbers 2, "The Mayan Numbers 3," "The Mayan Numbers 4," "The Egyptian Numbers 1," "The Egyptian Numbers 2," "The Egyptian Numbers 3," "The Egyptian Numbers 4," "Race Between Systems," "Comparing Systems," "Ancient Calculations," "The Use of the Abacus," "Let's Know the Abacus," "Decimals What a Passion," "Laura Don't Get Distracted."

In addition, there are 22 comics related to important mathematicians throughout history in the collection "Mathematicians in Comics," which can be downloaded free online or purchased in hard copy published by Daedalus Publishing House. In particular, for an in-depth look at the birth of our numbering system see the comic strip by mathematician Al-Khwārizmī (9th cent.).

Also available for active middle school teachers are teaching materials called "Mathematics in History," supplemented by student worksheets that can be downloaded from the ScuolaLab portal (where you must register to download the documents).

The following storybooks, suitable as early as elementary school, focus on counting and can be related to the use of the abacus and other counting tools:
- Abedi, I. (2002). *One, two, three... 99 sheep!* Daisy Editions.
- Bellei, M. (2020). *Cities of numbers*. Fatatrac.
- Cerasoli, A. (2012). *The great invention of Bubal*. Emme Editions.
- Cerasoli, A. (2019). *The five-fingered sisters*. Science editorial.
- Chermayeff, I. (2014). *Blind mice and other numbers*. Corraini.

- D'Angelo, S. (2008). *Never count on mice*. Topipittori.
- Fromental, J. L., & Jolivet, J. (2010). *10 little penguins*. The Beaver.
- Giusti, A. (2011). *Awa teaches counting.* The garden of Archimedes.
- Ohmura, T. (2011). *Everyone in the queue!* Babalibri.
- Tolstoy, A. (1999). *The giant turnip*. Fabbri.
- Urberuaga, E. (2015). *A black thing*. Lapis.

From this point of view, the rich collection of reviews "100 illustrated books between Italian and mathematics: a bibliography with teaching cues" written Demartini and Sbaragli is noteworthy.

The following narrative books, suitable as early as elementary school, focus on the history of mathematics:

- Cerasoli, A. (2012). *The great invention of Bubal*. Emme Editions.
- Giusti, A. (2011). *Awa teaches counting.* The garden of Archimedes.
- Petti, R. (2008). *Uri, the little Sumerian*. The garden of Archimedes.
- Petti, R. (2008). *Ahmose and the 999'999 lapis lazuli.* The garden of Archimedes.

In addition, for in-depth historical and educational study aimed at adults, the following references are recommended:

- Boyer, C.B. (1982). *History of mathematics*. Mondadori.
- Bunt, L., Jones, P. S., & Bedient, J. D. (1987). *The historical roots of elementary mathematics*. Zanichelli.
- Ifrah, G. (1989). *Universal history of numbers*. Arnoldo Mondadori. [Original ed. in French: 1981).
- D'Ambrosio, U. (2002). *Ethnomathematics*. Pythagoras.

D'Amore, B., & Sbaragli, S. (2017). *Mathematics and its history. I. From origins to the Greek miracle*. Daedalus editions.

- D'Amore B., & Sbaragli S. (2018). *Mathematics and its history: from the Greek sunset to the Middle Ages*. Daedalus editions.
- Fontana Bollini, V., & Lepori, G. (2019). A history of mathematics in middle school: the quadrature of plane figures. *Didactics Of Mathematics. From Research to Classroom Practices,* (6), 131-150.
- Nicosia, G. G. (2008). *Numbers and cultures*. Erickson.
- Vecchi, N. (2010). *How numbers came into being. An educational journey from primitive men to the abacus*. Rome: Carocci.

Also reposted below are some articles where the abacus is used as a tool. Some bring it up as an example, others compare it with other artifacts such as the pascaline.

- Bartolini-Bussi, M. G., & Mariotti, M. A. (2008). Semiotic mediation in the mathematics classroom: Artifacts and signs after a Vygotskian perspective. *Handbook of international research in mathematics education*, *746*.
- Bartolini-Bussi, M. G., & Boni, M. (2003). Instruments for semiotic mediation in primary school classrooms. *For the learning of mathematics*, *23*(2), 15-22.
- Maschietto, M. (2013). Systems of instruments for place value and arithmetical operations: An exploratory study with the Pascaline. *Education*, *3*, 221-230.

THE REMARKABLE NUMBERS

NEPER NUMBER "e"

Nepero's number, often denoted by the letter "e," is one of the most important mathematical constants and is used in several areas of mathematics, science and engineering. Its approximate value is about 2.71828. Nepero's number is an irrational and transcendent constant, which means that it cannot be represented as a division of integers and is not a solution of any algebraic equation with integer coefficients. This makes it a very special number in number theory.

Nepero's number is defined by the infinite series:

$$e = 1 + 1/1! + 1/2! + 1/3! + 1/4! + 1/5! + \ldots$$

Where "!" (factorial) represents the product of all positive integers up to the specified number. For example, $5!$ (5 factorial) is equal to. $5 \times 4 \times 3 \times 2 \times 1$.

Nepero's number naturally emerges in various mathematical contexts, such as differential and integral calculus, complex analysis, differential equations, probability theory, and exponential calculus. It is widely used in exponential growth and decay calculations, as well as in continuous interest evaluation in financial mathematics.

The name "Nepero's number" comes from the surname of Swiss mathematician Leonhard Euler, who frequently used the nickname "Nepero." However, the use of the name "and" to represent this constant was introduced by English mathematician Charles Maclaurin in 1718.

PI GRECO "π "

The number π (pi) is a fundamental mathematical constant that represents the ratio of the circumference of a circle to its diameter. It is an irrational constant, which means that its decimal value cannot be accurately expressed as a fraction and has an infinite, non-periodic representation. The symbol "π" is derived from the Greek letter "pi" (π), which is the first letter of the Greek word "perimetros," meaning "circumference."
The approximate value of π is 3.14159265358979323846... and so on, but there is no exact representation in the form of a fraction. The fraction $\frac{22}{7}$ is often used as an approximation of π, but it is only an approximation and is not exact.
The number π is used in many areas of mathematics and science, including geometry, trigonometry, calculus, and physics. It is one of the most important mathematical constants and

appears in many fundamental formulas and relationships. For example, in the area of a circle, the area A is given by $A = \pi r^2$, where r is the radius of the circle.
There are memorization competitions for memorizing the decimal digits of π, in which people try to remember and recite as many decimal digits as possible. The world record for memorizing the decimal digits of pi is about 70,000!
With the advent of computers, it has been possible to calculate billions of decimal digits of π, and this activity has been performed by many enthusiasts and researchers. However, for most practical applications, a few decimal digits of π are sufficient.

Feast of Pi: March 14 (3/14 in month/day format in the United States) is often celebrated as "Pi Day" because the date corresponds to the first three digits of π (3.14). Because of the similarity in the pronunciation of the words "pi" and "pie" (meaning "cake" in English), it is common to see pictures of celebratory cakes on Pi Day.

Encryption Ratio: The number π is often used in cryptography and computer security to generate random sequences and to create encryption algorithms.

ZERO

The "zero number" is an integer representing the absence of value or the null quantity. It is a fundamental concept in mathematics. The number zero is the only number that is neither positive nor negative and plays a crucial role in the number system and mathematical calculations.
The concept of zero has a history going back to many ancient cultures, including the Babylonians and Egyptians, who had symbols to represent emptiness or absence. The Maya also had a symbol for zero, used in a positional sense. However, the positional numbering system with zero as we know it today was developed primarily in India. Indian mathematicians, including Brahmagupta and Aryabhata, developed the concept of zero as a number and as a positional sign in the Indian number system. They introduced the concept of "sunya" or "shunya," meaning "emptiness," and began using the symbol "0" to represent this concept.

Zero was then transmitted to the West through Arab culture and trade. Arab mathematicians adopted the Indian system and introduced it to medieval Europe. Initially, the concept of zero was treated with skepticism in Europe. Some European mathematicians saw it as "nothing" and not as a real number. It took some time before the concept was fully accepted.
The introduction of the number zero and the positional number system revolutionized calculus and algebra, making calculations easier and enabling the development of more powerful methods for solving equations and performing mathematical operations.

In addition to being a number, the concept of zero has taken on a broader meaning in philosophy and number theory, often representing the idea of emptiness, absence or beginning. In engineering and science, the concept of zero is essential for representing energy levels, temperatures and other physical quantities. It is also essential for representing binary systems in computers.

In mathematical operations unolo zero is the neutral element in the sum. Adding zero to any number will not change the value of the number. For example, $5 + 0 = 5$. In addition, any number raised to the power of zero will remain equal to itself $n^0 = 1$.

ONE

The number one is the simplest of the integers and is the fundamental unit in the number system. It is the first and smallest positive integer and represents the idea of singularity, individuality and uniqueness. The mathematical symbol for the number one is "1," which is derived from ancient Roman notation and at the same time is reminiscent of ancient notches in wood.
In mathematical operations one is the neutral element in multiplication. Multiplying any number by one will not change the value of the number. For example, $5 \times 1 = 5$. In addition, any number raised to the power of one will remain equal to itself$n^1 = n$ and also one raised to the power of any number will remain one $1^n = 1$.

In the binary (base 2) system used in computers, any number is represented by a sequence of "1" and "0" bits. This bit is essential for representing digital data.

The number one is not considered a prime number, since prime numbers must have exactly two distinct positive divisors. The number one has only one divisor: itself.

INFINITE

In mathematics, the symbol ∞, called the "infinity symbol," looks like an upside-down number 8 and represents a very interesting concept: the idea of infinity.
For example, think of natural numbers. Imagine listing them: 0, 1, 2, 3, 4 and so on. No matter how far we go, we will never run out of numbers, will we? After a number by adding 1 we always get a next number. This is kind of like the infinity symbol. It tells us that we can keep counting and go on forever.

The infinity symbol represents precisely this idea of "infinity of elements," of "without end."

Besides counting, the infinity symbol can be used in many other situations. In more advanced math problems, you will see it in limits, which are like saying "what happens when something gets closer and closer to infinity?" Even when we talk about curves approaching a line without ever touching it, we can use the infinity symbol to represent this concept. Thus, the infinity symbol describes something that continues without ever ending. It is a bit like thinking of the horizon that extends to infinity, or as we have already seen the sequence of numbers that continues forever.

When you see the infinity symbol, you are thinking of something that is endless. It is a bit like an open door to a world of possibilities that has no boundaries!

Educational insights

The numbers and of Neperus and *pi*, being irrational, and the complex topic of infinity, are not explicitly included in the "Ticino Compulsory School Curriculum" as far as elementary school is concerned, but are "remarkable numbers" that are dealt with in later school levels. Instead, since elementary school the numbers 0 and 1 are dealt with in depth, both as elements of the set of natural numbers, emphasizing for 0 the fundamental role in our positional decimal system, and as neutral elements of some operations.

By proposing activities related to "remarkable numbers," several purposes can be pursued:

- grasp the characteristics of specific elements of our number system;
- Understand more deeply the structure of our positional decimal number system;
- learn about some important mathematicians of the past who dealt with these extraordinary numbers, their discoveries, and their relevance from a scientific and historical perspective;
- Knowing how to delve into a specific mathematical topic.

As for elementary school, the topic of infinity and the particularities of the numbers 0 and 1 within our number system are considered among the materials of the "MaMa-mathematics for elementary school" project. These materials are free to download at this link: https://mama.edu.ti.ch/.

In particular, it is suggested to consult:

- the *Guidelines* for having mathematical, educational and historical insights into the set of natural numbers, the role of infinity, and the role of the numbers 0 and 1 in operations;
- the *Teaching Practice* "Activities between mathematics and language in the second cycle" in which the topic of cryptography can be explored;
- the *Teaching Sheets* designed for learners, which you can find using the filter system in the search engine of teaching materials. Of particular note are "The Zero in Plus and Minus," "The Role of Zero," "Zero in Division," "The 0 in Decimals," "Even or Odd?", "Natural or Even?", "Let's Talk About Successions," and "Multiples and Divisors of 1."

The 0 is the protagonist of some nursery rhymes, songs and games related to the project "A spasso con i numeri naturali" curated by the Mathematics Didactics Center of the Department of Formation Learning / Alta scuola pedagogica of SUPSI in Locarno together with RSI KIDS, soon to be released online.

To learn more about the number *pi*, you can view the entertaining video "Televendita del π" from the "Matematicando Ciak!" series, which consists of a collection of videos for educational use. In the video "Finding Zero," however, it is precisely the first natural number that is the protagonist. To have fun to the rhythm of music on the theme of infinity, you can listen to the song "Do you know what infinity is?"
To learn more about the number pi *and* the number and in history and identify activities aimed at high school, you can check out Matabel's activity "How real are transcendent numbers?" or the interesting podcast on *pi* from the "Fantamathematics" review (almost true stories of mathematicians and other people with problems.

Regarding cryptography, it is possible to download the poster "How to protect information?" linked to the screening of the film "The Theory of Everything," shown as part of the Mathematics film festival.

In the collection "Mathematicians in Comics," which can be downloaded for free or purchased in hard copy published by Daedalus Publishing House, there are 22 stories related to important mathematicians in history. For an in-depth look at the mathematician, physicist and astronomer Leonhard Euler, you can consult the comic strip Euler (18th cent.), where a famous theorem of graph theory is addressed in a popular way. For an in-depth look at some historical discoveries related to infinity, you can download Cantor's comic strip (19th cent.).

Bibliographic references for educational use:
- Cerasoli, A. (2011). *The magnificent ten*. Editorial Science.
- Cerasoli, A. (2011). *The adventures of Mr. 1.* Emme Editions.
- Cerasoli, A. (2015). *All in celebration with pi*. Science Publishing.
- Feniello, A. (2014). *The child who invented zero*. Laterza publishers.
- Melis, A. (2007). *Prince Zero*. Piemme editions.
- Novelli, L. (2015). *Hello, it's zero*. Valentina Editions
- Rittaud, B. (2014). *1, 2, 3... infinity!* Daedalus editions.
- Rodari, G. (1980). *The triumph of the zero*. EL editions.

In addition, for in-depth historical and educational study aimed at adults, the following references are recommended:
- Arrigo, G., D'Amore, B. & Sbaragli, S. (2020). *Mathematical infinity. History, epistemology and didactics of a fascinating topic.* Pythagoras.

- D'Amore, B., Asenova, M., Del Zozzo, A., Fandiño Pinilla, M. I., Iori, M., Nicosia, G. G., & Santi, G. (2021). *Numbers. Mathematics, history, games, and trivia for correct and effective teaching*. Pythagoras.
- D'Amore, B. & Fandiño Pinilla, M. I. (2009). *Zero*. Erickson.
- García del Cid, L. (2015). *Remarkable numbers*. RBA.
- Odifreddi, P. (2020). *Portraits of the infinite*. Rizzoli.
- Villani, V. (2003). *Starting from scratch.* Pythagoras.

DO ANIMALS MATTER?

This section is loosely inspired and taken from this link to which we refer for further discussion:
https://alessadra.wordpress.com/2010/01/18/animali-matematici-2/

The crow who could count to 5
(from a story from the 1700s)

A farmer was trying to kill a crow that was nesting on top of a tower on his land, but each time he approached, the crow flew away and returned only when the farmer had moved away by entering the house. The farmer asked a neighbor for help, but the crow always evaded them by coming out only when they were both had returned to the house. As the number of peasants increased, the crow always waited for everyone to re-enter the house before returning to the nest. Only when five peasants re-entered the house and one remained outside did the crow return to the nest and was killed.

This raises the question of whether the crow could only count to five. Studies conducted by ethologist Otto Koehler sixty years ago suggest that some birds, such as Jacob, a crow, can associate the number of dots on a lid with the number of dots on a card and distinguish between numbers from 2 to 6. Animals, including birds, have the ability to compare quantities and remember numbers at successive times. According to cognitive psychologist Stanislas Dehaene, both animals and humans have an intuitive representation of quantities, although they do not count as exactly as we do.

Turkeys, chicks and other mathematical birds

Not only crows, but also other bird species demonstrate mathematical skills. Otto Koehler trained turkeys to lift box lids to obtain a specific number of pieces of food, stopping when they reached the desired amount. Canaries were also trained to choose the fifth tablet between communicating cages, demonstrating numerical skills. African parrot Alex, trained by psychologist Irene Pepperberg, learned extensive vocabulary, including numbers one through six, and correctly answered questions about the number of objects in a tray, even though he had never seen them before.

Recent studies conducted by Giorgio Vallortigara have revealed that chicks are also able to count to five and organize numbers in ascending order from left to right, similar to what is done by humans. Birds thus demonstrate an amazing ability to understand and use number concepts.

Can lions count?

The experiment conducted by Karen McComb in Tanzania's Serengeti Park involved a lioness who, upon hearing unfamiliar roars, inferred the presence of intruders in her territory. Initially, when she heard a single roar, the lioness proceeded to search for her group, fearing a confrontation as equals. Later, hearing a chorus of roars, she inferred the presence of three intruders but was in the company of four other lionesses in her group, bringing their total to five against three.

The leading lioness approached the point of origin of the roars and, together with the others, rushed into the trees. However, they found no intruder; the roars came from a speaker used in McComb's experiment. Brian Butterworth, a professor of neuropsychology at University College London, suggested that the lioness leader's behavior could be explained by enumerating the roars heard and the members of her group, then comparing the two numbers. This demonstrates the ability of the lioness to abstract the numerosity of intruder and defender sets regardless of the sensory modality by which she perceives them, underscoring their remarkable skill in numerical cognition.

Cunning rats

Experiments in the 1950s and 1960s showed that rats also possess number perception. In one experiment, rats were placed in a cage with two buttons, "A" and "B," and in order to get a small ration of food they had to press button "A" a certain number of times before switching to button "B." If they made a mistake in the prescribed sequence, they were given a penalty, such as a slight electric shock or turning off the light. Initially, the rats learned to press "A" several times and "B" only once, but later they were able to refine the number of times they had to press "A" according to the number "n" set by the trainer.

The number "n" was not always accurate, but the rats responded in an approximate way, for example, if they had to press "A" 4 times, they might happen to press 3 to 7 times. The *experiment was further refined by introducing a loudspeaker that emitted sequences of sounds, thus confirming the rats' ability to recognize approximately the number of objects, sounds or portions of food. This demonstrates their ability to perceive and use approximate numerical concepts.*

The math skills of our chimpanzee cousins.

Chimpanzees, through numerous experiments, have demonstrated competence in basic arithmetic. A notable example is Ai, trained at the Kyoto University Primate Research Institute, who can recognize Arabic numerals from 0 to 9 associated with objects and sort them in increasing or decreasing order. Sheba, another chimpanzee, outperformed Ai after extensive training, demonstrating the ability to sum objects and indicate the result using abstract number symbols instead of concrete objects.

Kanzi, a bonobo, is a "genius" among apes, residing at the University of Georgia's Language Research Center in the United States. He has learned more than 100 terms by observing human language teaching efforts to his mother, Matata. Kanzi now communicates with researchers, even remotely via telephone, and recognizes abstract concepts such as good and evil, demonstrating a remarkable ability to understand language and concepts in addition to concrete ones. These studies highlight the amazing cognitive and linguistic abilities of chimpanzees and bonobos.

Circus phenomena

"Wise" animals trained to perform in circuses and theaters have always attracted interest, often through tricks with trainers. A well-known example is Hans the Clever, a German horse trained by Wilhelm von Osten, a mathematics teacher. Hans seemed able to solve math problems and spell words. During performances, the audience would ask questions, such as "How much is 4 + 6?" and Hans would answer by hitting the ground with a hoof the correct number of times.

In 1904, a commission of inquiry chaired by Carl Stumpf was convened to examine Hans' abilities. After a thorough analysis, they concluded that there were no tricks in his performance. However, Oskar Pfungst, a student of the committee chairman, demonstrated that Hans was receiving signals from people in the audience or from von

Osten indicating when to stop beating his hoof. These signals were involuntary, such as a movement of the eyelashes or a change in the tension of the questioner as the horse approached the right answer. In fact, Hans had no mathematical skills, but he was sensitive to involuntary nonverbal cues that told him when to stop. This case highlights the importance of thorough scientific investigations to unravel the apparent numerical ability of trained animals.

Conclusions

The potential and limits of mathematical intelligence in animals remain a question without a definitive answer. The study of animal intelligence is still at a rough stage, but newly available investigative tools make it possible to explore brain activity and neural circuits related to mathematics, although they are still at an early stage for animals compared to humans.

According to Stanislas Dehaene, humans share with animals such as mice, pigeons and monkeys a mental representation of quantities that allows them to quickly number sets of objects, perform addition operations and compare cardinalities, all without the need for language. These abilities, inherited from evolution, enable estimation of the size of a set and can also influence the understanding of numbers expressed symbolically, as in Arabic numerals. In summary, the intuition of numerical quantities inherited from evolution could play a key role in the development of advanced mathematics in humans. However, many questions remain unanswered, and research continues to explore the mathematical capabilities of animals.

Educational insights

The topic of animal numerical skills allows awareness to be raised toward different ends:

- Understand that human beings are not the sole repositories of knowledge;
- To grasp connections with the surrounding environment, particularly the animal world;
- Opening the gaze to interdisciplinarity.

A reference suitable for adults that is useful for further study is as follows:

- Bagini, B. & Dulio, P. (2016). *Mathematics for rabbits*. TAM Publisher.
- D'Amore, B. (2007). *Mathematics everywhere*. Pythagoras.

HOW MANY ARE THERE?

THE WISDOM OF THE CROWD

THIS LOCATION MAY INDIFFERENTLY BE IN THE FINGER OR DATA SECTION since the idea is to count how many objects are in the plexiglass containers but since it is difficult to count them we estimate their quantity.

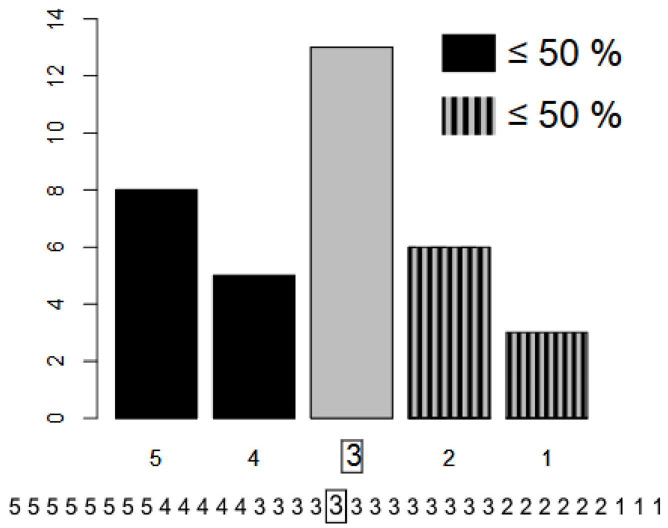
What is it. How many candies are in the glass jar? Mathematics tells us that we find the best estimate by appealing to what scholars call "The Intelligence of the Crowd" (or the wisdom of the crowd), a sociological theory that a collection of people can make any estimate better than experts can.

Why? In 1906, anthropologist and statistician Francis Galton, cousin of the famous Charles Darwin, asked the audience at a cattle fair to estimate the weight of an ox. He collected all the estimates and noticed that the median -- put all the values in ascending order, the median being the middle value -- of the answers given by the crowd came closer to reality than the answers given individually by the experts present.
In mathematical terms, the median of a data set corresponds to the value that lies exactly in the middle in the ordered sequence (ascending or descending) of the data. A histogram for a data set appears in the figure below. The data values are:

5,5,5,5,5,5,5,5,4,4,4,4,4,3,3,3,3,3,3,3,3,3,3,3,3,3,2,2,2,2,2,2,1,1,1 and correspond to the number of members in 34 families: some families consist of only one person, specifically there are 3 such families. Then there are families with 2, 3, 4 up to 5 members.

The first step is to sort the data, and in this case we have already sorted them in descending order.

The second step is to find the value that occupies the middle position. This value corresponds to the median, 3 in the example.



Fact: The crowd wisdom theory finds great application on the Internet and especially on sites such as *Yahoo!* Answers and *Wikipedia,* which base their operation on it and rely on the content generated by the large number of experts who contribute to the creation of *Wikipedia* pages or answers.

Teaching links: ***Basics of statistics (AVERAGE, AVERAGE, MODE). See the chapter on "Estimates" for further study.***

External resources:

(In English) https://youtu.be/n98BhnwWmsc
The wisdom of the crowd put into practice for a contest.

(In English) https://www.scienceathome.org/
This project proposes games based on open problems, e.g., quantum physics, with the intention of collecting data on users' game sessions to carry out real scientific research.

(In English) https://www.zooniverse.org/

Like the previous one, but without the game guise. Users can easily contribute to scientific or public benefit studies (such as highlighting damaged areas on satellite images after the earthquake in Ecuador).

Educational insights

Proposing experiences related to different types of *estimation*, turns out to be very meaningful as early as elementary school. The main types of estimation concern *numerosity* (i.e., numerical quantities), *measurements* (of both continuous and discrete quantities) and *computational* aspects (referring to the results of calculations). The candy example above deals with numerosity estimation, which concerns the recognition of quantities greater than 5-7 elements.

As reported in this section, it is possible to link estimation experiences to visualizations of quantities using graphs and tables, another topic covered as early as elementary school.

Such experiences enable the development of different purposes:

- Knowing how to train the "eye for esteem," both direct and indirect;
- Predict the order of magnitude of a quantity, a magnitude, a calculation;
- Being able to accept the presence of an error in one's estimate, compared to the exact value;
- Knowing how to use and coordinate various mental calculation strategies with each other;
- Knowing how to represent a quantity through different forms of representation.

As for elementary school, some ideas related to estimation are included in the materials of the "MaMa-mathematics for elementary school" project. These materials are free to download at this link: https://mama.edu.ti.ch/.

In particular, it is suggested to consult:

- the *Guidelines* for having general mathematical and educational insights related to estimation in mathematics;
- the *Teaching Practices* "Let's Estimate How Many," an insightful document aimed at the first cycle, and "Let's Estimate Quantities and Outcomes," a document aimed instead at the second cycle, but also suitable for older learners.
- the *Teaching Sheets* designed for learners, which can be found by setting the "Estimate" filter in the teaching materials search engine. Among the many cards present, those that are specifically suited for work similar to that presented in this paper are "Containers and Estimation," "Estimating in Pairs," "Fruit at a Glance," "How Many Candies," "At the Zoo", "Stars in the Sky," "Comparing Estimates," "How Many Staples," "How Many Chickpeas."

From the same portal it is also possible to retrieve many materials related to the use and reading of graphs and tables. In particular, it is suggested to consult:

- the *Guidelines* to have mathematical and educational insights related to the use and implementation of graphs and tables, as well as some historical nods to the topic;

- the *Teaching Practices* "Let's Play with Graphs and Tables," a document full of activity ideas for introducing reading and making graphs and tables in the classroom;
- the *Teaching Sheets* designed for learners, which can be found by setting the filter "Charts and Tables" in the teaching materials search engine. Among the many cards present, those that are particularly suitable for work similar to that presented in this paper on histograms are the following: "Class 1 Charts," "Class 2 Charts," "Weather Problems," "Class Register," "All on Vacation," "Choosing a Movie," "Recreations," "Class Survey," "From Table to Chart," "Female Names," "Podium of Female Names," "Age of Population," "What Languages Do You Speak," "Numbers and Families," "The Verbanella," "School Attendance," and "Fruit and Vegetable Garden."

By consulting the teaching sheet "We estimate with the senses" in the "Mathematics" (matematicando.supsi.ch), suitable for students in the first cycle of elementary school, you can get teaching cues to play on estimating quantities not only through sight. For the second cycle of elementary school, additional cues can be found in the "Let's Estimate ourselves" and "Estimating in the Square" teaching sheets. A more complex version of the last teaching sheet, suitable for middle school, is available under the same name "Estimates in the Square."

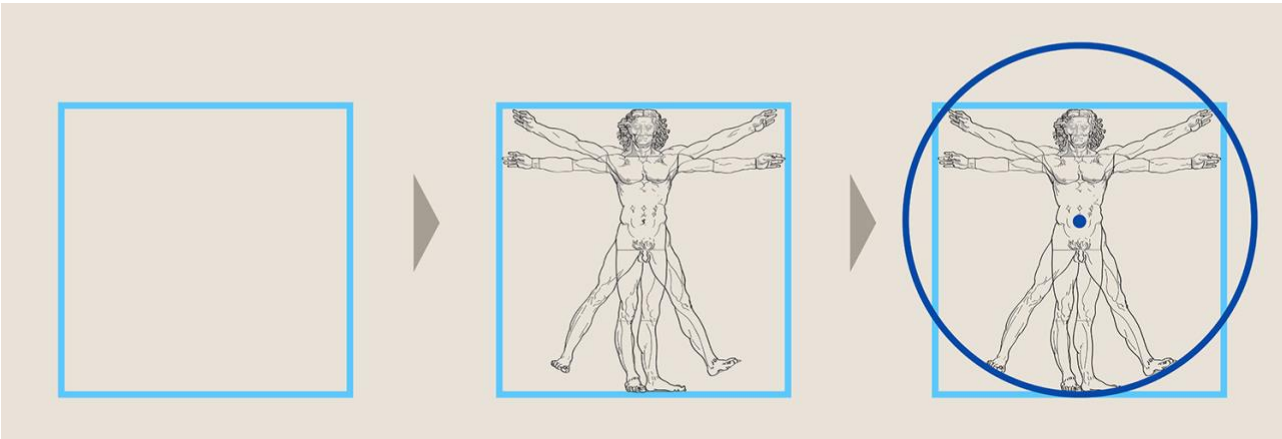
The following texts, suitable as early as elementary school, focus on displaying large quantities of elements. They can then be used didactically by showing pupils the illustrations and asking them to try to make an estimate of the quantity of elements represented, then verifying by counting (for small quantities) or reading the numbers directly on the page (for larger quantities).

- Tessaro, G. (2012). Many many *many*. Carthusia.
- Fromental, J. L., & Jolivet, J. (2017). *365 penguins*. The Beaver.
- Roskifte, K. (2019). *Everyone matters*. EL editions.

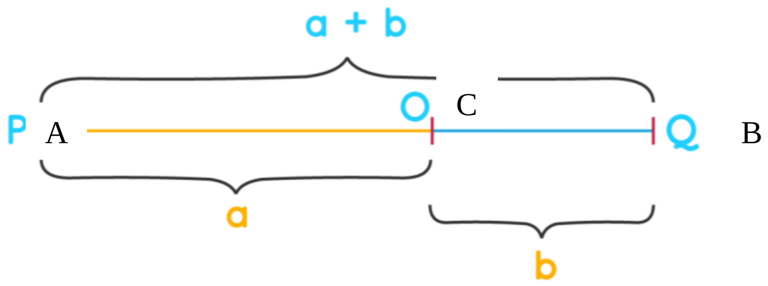
AUREAN NUMBER

What it is. The Vitruvian Man is a drawing by Leonardo da Vinci circa 1490, which depicts a man in two overlapping positions within a circle and a square. The purpose of the drawing was to represent the ideal proportions of a body and to demonstrate how it can be inscribed in the circle and square, the two "perfect" geometric figures. The work is so named because it was inspired by the work of the Greek architect and philosopher Marcus Vitruvius Pollonius, who in 15 B.C. thus wrote:

> "*Moreover, the center of the human body is by nature the navel; in fact, if you lie a man on his back, hands and feet spread wide, and point a compass at his navel, you will touch tangentially, describing a circle, the ends of the fingers of his hands and feet.*"



Why. The perfect proportion (or golden section) of the body within a circle can be represented by a segment divided into two parts a and b, where $a+b$ represents the total height of the person and a the distance from the ground to the navel, such that the ratio of $a+b$ to a is equal to the ratio of a to b. From which we get that the golden section is: $1,618033988$.
Concretely, to have perfect proportions, a tall person $1.62m$ will have to have the navel about $1m$ from the ground.



$$\frac{a}{b} = \frac{a+b}{a} = 1.618.... = \phi$$

Educational Links: ***How to Calculate*** φ?

The golden section of segment AB is segment AC, with C between A and B, the proportional mean between the entire segment AB and the remaining part CB, i.e.

$$AB\!:AC \;=\; AC\!:CB$$
$$AB \,:\, AC \;=\; AC \,:\, (AB - AC)$$

We indicate $AB = x$
e $AC = a$

$$x : a = a : (x - a)$$
$$a^2 = x \cdot (x - a)$$
$$x^2 - ax - a^2 = 0$$

This is a second-degree equation in the unknown x, which admits two solutions only one acceptable because it is positive having the value:

$$x_1 = a\frac{1+\sqrt{5}}{2}$$

At this point we define the ***golden*** ratio as the ratio $\varphi = \frac{x}{a}$ ie:

$$\varphi = \frac{1+\sqrt{5}}{2}$$

What is so special about it φ?

- The square of $\varphi$is equal to φincreased by 1: $\varphi^2 = \varphi + 1$
- The reciprocal of $\varphi$is equal to φdecreased by 1: $\frac{1}{\varphi} = \varphi - 1$

- The properties we have just outlined indicate, among other things, that the square and the reciprocal of φhave the same decimal part as φ:

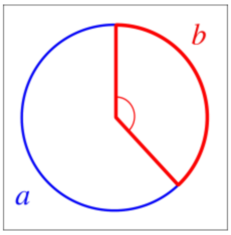
$$\varphi = 1{,}6180339887 \dots$$

$$\varphi^2 = 2{,}6180339887 \dots$$

$$\frac{1}{\varphi} = 0{,}6180339887 \dots$$

  In addition, φ is the only number for which this situation (if we exclude natural numbers, of course).

In geometry, the **golden angle** is the angle subtended by the smallest arc of the circumference (the arc in red in the adjacent figure) that is obtained by dividing the circumference itself into two arcs that stand between them in the same ratio as in the golden section.
The value of this angle is $137^{\circ}\,30'$.

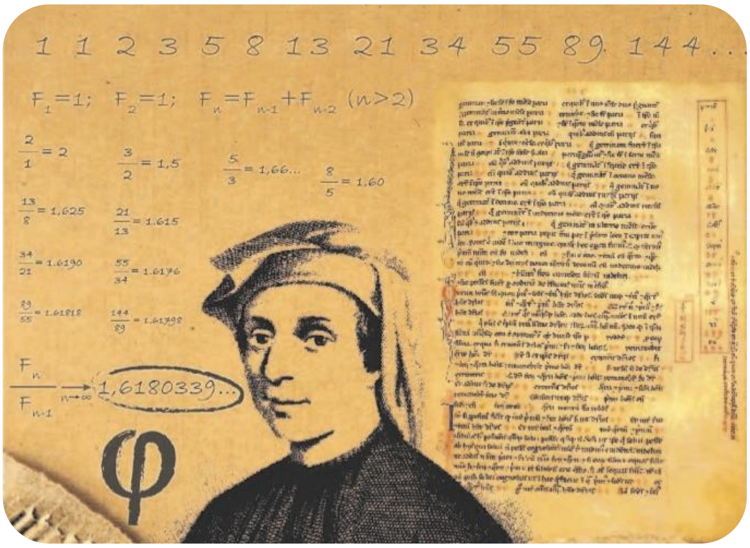


Curiosity**:** For us it is just a nice curiosity, but for centuries, we have been finding the golden ratio behind the idea of harmony and perfection. Moreover, there is no scientific evidence that those proportions are "more beautiful" than others, so it remains just a matter of taste and culture.

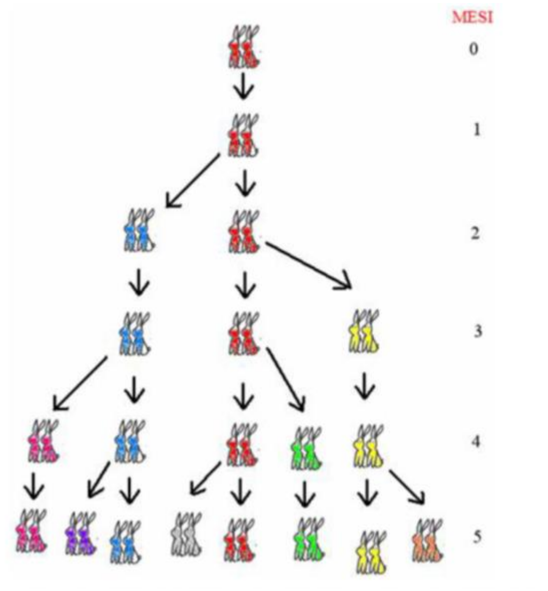
Other external resources:
http://wsimag.com/it/cultura/2004-le-divine-proporzioni
Article by Piergiorgio Oddifreddi on the proportions that inspired art.

MATHEMATICS AND NATURE

THE FIBONACCI SUCCESSION



$1, 1, 2, 3, 5, 8, 13,\ 21, 34,$ ... This succession, in which each number is obtained by the sum of the two numbers that precede it, is named after Leonardo Pisano known as Fibonacci (1175-1240), a 13th-century mathematician. His father was a merchant who had constant contact with the peoples of North Africa, so Leonardo had the opportunity to travel to Egypt, Syria, and Greece and to meet leading Arab mathematicians and learn algebra and the Indo-Arabic notation system that has come down to us. In 1202 he wrote the Liber Abaci, a work that introduced Europe to the Indo-Arabic system of notation, decimal numeration and the zero-or zefr, from the Arabic zefiro-that is, empty number like the "breath of wind," which, by moving digits as the wind moves things, changes their value even in its insubstantiality.



Fibonacci succession arose from a Liber Abaci problem: "*How many pairs of rabbits do you get in a year, barring death, assuming that each pair gives birth to another pair every month and that the youngest pairs are able to reproduce as early as the second month of life?*"

The answer is as follows: at the end of the first month you have the first pair; at the end of the second month you have the original pair and a new pair generated by it; at the end of the third month a third pair is added; at the end of the fourth month you have 5 pairs, because the second pair has also begun to generate, and so on:

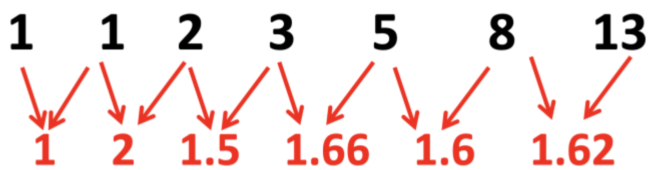
$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, 10946, 17711 \ldots$

The Fibonacci succession possesses many elegant and significant properties. Let's look at some of them:

- Two consecutive Fibonacci numbers have no common factors, that is, they are prime to each other (coprime), in other words, their greatest common divisor is equal to 1.
- Any number in the succession raised to the square is equal to the product of the number before it and the number after it, increased or decreased by one unit.

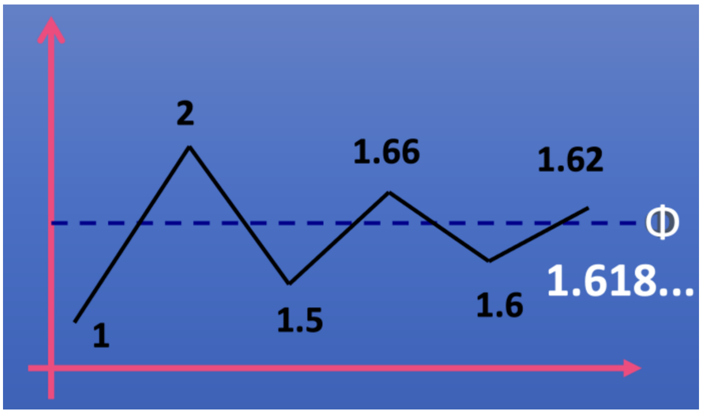
For example. $21^2 = 441 = 13 \times 34 - 1$
while $89^2 = 7921 = 55 \times 144 + 1$

- The succession formed by the ratios between two consecutive terms of the Fibonacci succession is a succession whose first terms are:



$\frac{1}{1}$	$\frac{2}{1}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{8}{5}$	$\frac{13}{8}$	$\frac{21}{13}$	$\frac{34}{21}$	$\frac{55}{34}$	$\frac{89}{55}$	$\frac{144}{89}$	....
1	2	1,5	1,666	1,6	1,625	1,615	1,619	1,617	1,61818	1,6179	...

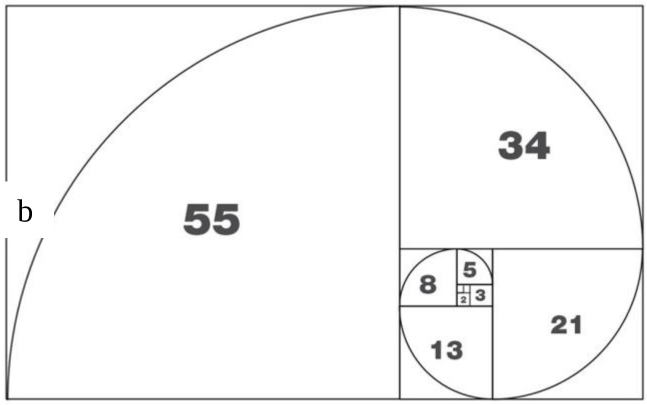
The limit of this succession is The "golden number" or "golden ratio"; it was known since antiquity for its curious characteristics and is denoted by the Greek letter ϕ (phi) in honor of Fibonacci. It is an irrational number: its decimal digits continue to infinity with no apparent pattern! The "golden ratio" was called "divine proportion" by Leonardo da Vinci.



Spirals and Fibonacci numbers

The numbers of the famous Fibonacci succession 1, 1, 2, 3, 5, 8, 13, 21 can be used to draw appropriate squares as in the figure: this set of rectangles, whose sides have lengths equal to successive Fibonacci numbers and which are composed of squares with sides that are Fibonacci numbers, are called Fibonacci rectangles. Drawing in each square a quarter of a circle yields the Fibonacci spiral, which closely approximates the spiral called the *golden spiral*: a special logarithmic spiral that increases by the golden number φ with each quarter turn.

a



$$\frac{a}{b} = \varphi = 1{,}618$$

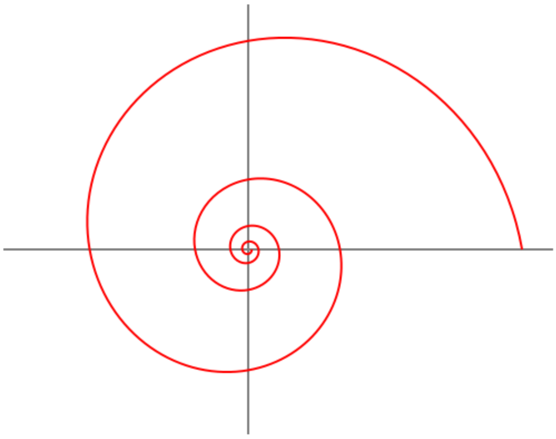
Spirals, however, are not all the same: depending on the law you choose, you can have Archimedes spirals, Galileo spirals, Fermat spirals, logarithmic spirals, hyperbolic spirals and several others. How to mathematically draw a spiral? Drawing with a pencil from a central point, simply move it by a distance *r* and rotate it at the same time by an angle θ, following a rule that modifies *r* as r changes. θ. Some parameters (*a*, *b*) help deform the spiral even though the rule remains the same.

logarithmic spiral ($r = ab\theta$)

Fermat's spiral ($r = a\sqrt{\theta}$)

Hyperbolic spiral ($r = a/\theta$)

Archimedes spiral ($r = a + b\theta$)



Logarithmic spiral

It seems that the logarithmic spiral serves as a model, albeit always in an approximate way, for many natural structures. It is a shape that has also inspired the imagination of artists and scientists for centuries because of its aesthetic and mathematical properties-so much so that Bernoulli called it *spira mirabilis* and wanted one engraved on his tombstone-unfortunately, the sculptor made one of Archimedes instead!

The spiral is one of the most fascinating structures that recur in the universe.

FIBONACCI NUMBERS IN NATURE

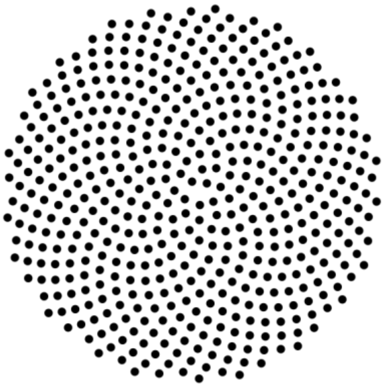
One of the special features of Fibonacci numbers is that they are often found in nature.

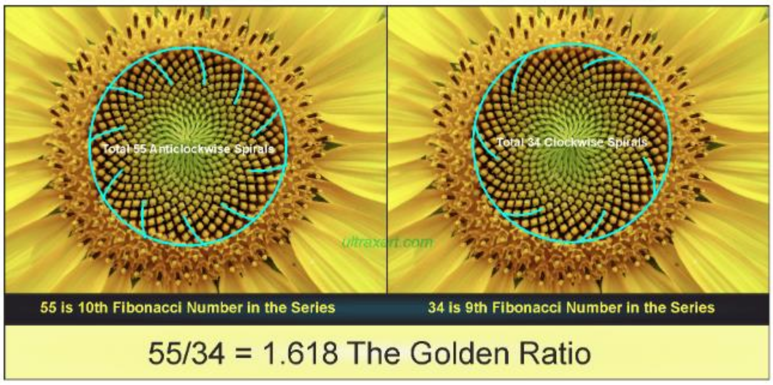
What makes this connection interesting is that by following the Fibonacci sequence, flowers can maximize the space and effectiveness of their exposure to sunlight and pollinating insects. An arrangement of petals or seeds that follows this sequence allows flowers to have optimal positioning to catch light and insects from different angles. For plants, carrying multiple seeds is advantageous because it increases the species' chances of reproducing.

However, it is important to note that this rule does not apply to all flowers, as there are many factors that influence the number of petals, including genetics, environment and other variables.

For example:

- In **sunflowers,** the central disk (the flower head) carries a multitude of primordia that, as they mature, will become little flowers and then seeds. But how are all these primordia arranged in such a small space? The first one (called the apex) is born in the center, the subsequent ones are each born rotated by a certain angle (golden angle of $137^\circ 30'$associated with the golden number) relative to the previous one. The seeds are arranged following a spiral structure. These spirals are related to Fibonacci numbers, In addition, the proportion of seeds arranged in these spirals follows the golden ratio ϕ. This means that the number of clockwise spirals is often a Fibonacci number, while the number of counterclockwise spirals is often the next Fibonacci number.

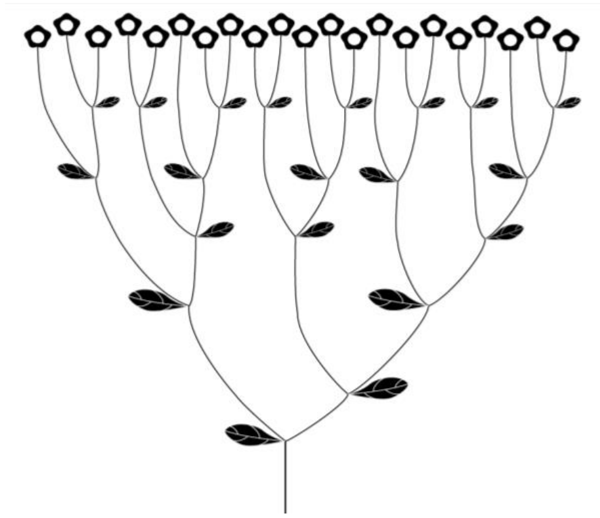




- **Daisies** are often cited as an example of flowers with petals that follow the Fibonacci sequence. They are known to have 21 or 34 petals.
- **Hyacinths** are bulbous flowers with an arrangement of 3, 5 or 8 petals.
- **Lilies** can have a petal arrangement that approximates the Fibonacci sequence, with 3, 5 or 8 petals.
- **Marigolds** are flowers often with 21 or 34 petals.
- The **buttercup** flower can have 5, 8 or 13 petals.
- Achillea Ptarmica (also called **sternutella**) is a plant that flowers in the summer producing many small white flowers, one per branch. As is often the case in nature, its growth seems to follow a definite pattern: one branch generates a new one, which after a month will be able to produce another, and so on. How many branches and thus little flowers will appear after one year? A simple mathematical scheme can help solve this problem.
  The first step is to draw a simplified diagram of the plant with the branches month by month, keeping in mind that a new branch will only be able to split after one month. The number of branches increases quickly, and the pattern soon becomes intricate. To understand what happens after twelve months, it can be seen that there is a rule in this pattern. In fact, the number of branches that arise each month is the sum of the branches in the previous two months:

$$21 = 13 + 8, \;\; 13 = 8 + 5, \;\; 8 = 5 + 3, \;\; 3 = 2 + 1$$

  The branch count brings out the Fibonacci succession.



- Spirals can also be seen in other **plants** such as cauliflowers, pine cones, and cacti.
- Nautilus shells, chameleon tails, galaxies, rolled leaves, snail shells, rose petals--spiral shapes are often encountered in nature.





- The *cochlea-from* Latin *cochlea*, cochlea-is a structure we have in our inner ear resembling a spiral. The right cochlea has a counterclockwise coil, while the left cochlea has a clockwise coil.

Teaching links: ***Irrational numbers***.

It is remarkable that many of the most important numbers in mathematics are irrational (π, , and, φ) that is, numbers with infinite decimal digits that continue without repeating. Much can be said about their properties; here it suffices to point out that the definition of φ as "the most irrational of irrationals" derives from the difficulty of approximating it with *continuous fractions*. It is an alternative representation to the decimal representation, which is more effective for making approximations of any real number with rational numbers, that looks like this:

$$numero = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\dots\dots}}}$$

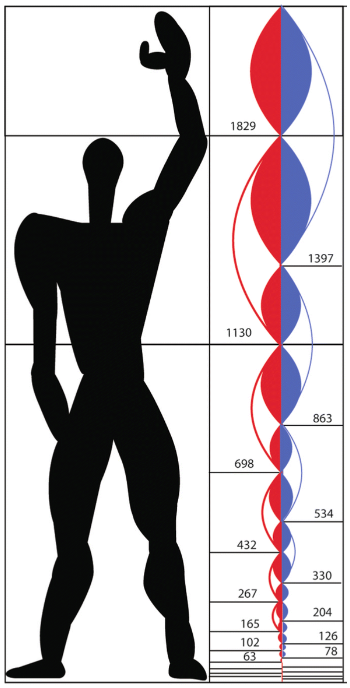
Where $a_{0,1,2,3,\dots}$ are all integers. The more you expand the fraction, the more the precision of the approximation improves, as if you used a finer ruler to represent the number. Any real number, so even irrational ones, can be represented in this way, for example:

$$\pi = 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{\dots\dots}}} \qquad \text{while} \qquad \varphi = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\dots\dots}}}$$

The presence of 1 everywhere in the representation of φ makes the accuracy grow very slowly, which makes it the worst to approximate with rational numbers.

Other external resources:
https://it.wikipedia.org/wiki/Frazione_continua
Continuous fractions on Wikipedia.
(In English) http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/
A veritable mine of information on the golden number and other mathematical curiosities.

Educational links: ***mathematics and art.***



In the past, the golden section has been thought to have inspired many architects of antiquity and in particular the builders of the Parthenon and Pyramids, as parts of these buildings correspond to the ratio expressed by this proportion. In truth, there is no evidence that the golden section was ever used on purpose in the construction of these monuments.

Closer to the present day, Swiss architect Le Corbusier (1887-1965) based part of his architectural work on the Fibonacci series and the golden section; he also created an architectural scale system called "Modulor." The term "Modulor" is a combination of the words "module" (module) and "d'or" (golden), recalling the concept of golden proportion (golden number) in mathematics. Le Corbusier's goal was to create a scale of measurements based on human proportions and the concept of mathematical harmony.This system was used to guide design decisions such as floor heights, furniture heights, window ratios and more. The theory was that following the Modulor would result in architectural spaces and designs that are pleasing and harmonious perceived by the person using them.The Modulor was applied in various architectural projects by Le Corbusier, such as "Modulor 1" and "Modulor 2," and had a significant impact on architectural design and design theory.

Educational links: ***mathematics and music theory.***

An extraordinary invention such as the Fibonacci succession finds wide application in disciplines other than mathematics, including music, where it is expressed as pure melodic nature. Assign each number a musical note: C = 1, D = 2, etc., and consider a diatonic scale of 7 notes: C-RE-MI-FA-SOL-LA-SI. Using the *modulo* operation, it is possible to adjust the seven natural notes to the infinite succession of numbers. With this operation, each number becomes the remainder of its division by 7.

E.g. $15 = 2 \times 7 + \mathbf{1}$, therefore 15 = **1** (*mod7*).

If we perform the operation on the n-th number *mod* 7 we obtain the results contained in the following table where the Fibonacci succession is shown in the first column and the corresponding number *mod* 7 in the second.

N. succ.	N. mod 7	N. succ.	N. mod 7	N. succ.	N. mod 7
1	1	17.711	1	433.494.437	5
1	1	28.657	6	701.408.733	4
2	2	46.368	0	1.134.903.170	2
3	3	75.025	6	1.836.311.903	6
5	5	121.393	6	2.971.215.073	1
8	1	196.418	5	4.807.526.976	0
13	6	317.811	4	7.778.742.049	1
21	0	514.229	2	12.586.269.025	1
34	6	832.040	6	20.365.011.074	2
55	6	1.346.269	1	32.951.280.099	3
89	5	2.178.309	0	53.316.291.173	5
144	4	3.524.578	1	86.267.571.272	1
233	2	5.702.887	1	139.583.862.445	6
377	6	9.227.465	2	225.851.433.717	0
610	1	14.930.352	3	365.435.296.162	6
987	0	24.157.817	5	591.286.729.879	6
1.597	1	39.088.169	1	956.722.026.041	5
2.584	1	63.245.986	6	1.548.008.755.920	4
4.181	2	102.334.155	0	2.504.730.781.961	2
6.765	3	165.580.141	6	4.052.739.537.881	6
10.946	5	267.914.296	6	6.557.470.319.842	1

This yields a sequence of numbers that vary between 0 and 6 and which, therefore, can be simply transformed into the seven musical notes as follows:

1	2	3	4	5	6	0
DO	KING	MI	FA	SOL	LA	YES

On the basis of this operation, it is possible to construct some chords that resonate harmonically with the spiral in our ear and evoke a pleasant experience, including notes whose position in the musical scale is associated with Fibonacci numbers, e.g. 3-5-8. For those numbers not directly associated with the diatonic scale, we proceed as follows.

The numbers of the succession after the modulo operation are given by:

1	1	2	3	5	1	6	0	6	6	5	4	2	6	1	0
DO	DO	KING	MI	SOL	DO	LA	YES	LA	LA	SOL	FA	KING	LA	DO	YES

By substituting recursively, we see that the musical sequence repeats identically every sixteen notes. Thus, one obtains a melody that can be played in only one octave, avoiding touching those notes that by swirling up would no longer be perceptible or intelligible.

Other external resources:
https://emercurius.wordpress.com/2011/09/12/fra-numeri-e-musica-2/
Informal insight into the "music" of the Fibonacci sequence, from which you can also listen to it.

Educational insights
Successions, particularly the well-known Fibonacci succession, are proposed as early as elementary school with the intention of developing the following purposes:
- Approaching the idea of recursiveness;
- get to know a well-known mathematician from history, so as to make mathematics more "human."
- Connect mathematics to other subject areas;
- Connect mathematics to real-world contexts.

The number φ or *golden number*, on the other hand, is not part of the elementary school curriculum, but it can in any case be presented at this school level in an intuitive way, placing it in various contexts, and then later deepening it mathematically in middle and high school.
Proposing the *golden number* in the education of pupils, in addition to developing the purposes already proposed in the section "Remarkable numbers," specifically enables:
- Deepen the mathematical characteristics of a particular type of number;
- Find connections between mathematics and other cultural areas;
- grasp mathematics in real-world contexts.

As for elementary school, some ideas related to successions (some related to the Fibonacci succession) are included in the materials of the "MaMa-mathematics for elementary school" project. These materials are free to download at this link: https://mama.edu.ti.ch/.
In particular, it is suggested to consult:
- the *Guidelines* for having general mathematical and educational insights related to the topic of successions;
- the *Teaching Practices "Discovering Successions,"* an insightful document aimed at the first cycle, and "Let's sharpen our wits with successions," a document aimed instead at the second cycle, but also suitable for older pupils.
- the *Teaching Sheets* designed for learners, which can be found by setting the filter "Successions" in the teaching materials search engine. Among the many cards present, the one that explicitly lends itself to proposing the rabbit problem, and thus discovering the Fibonacci succession, is called "The Rabbit Problem."

From the same portal, it is also possible to retrieve some materials related to interdisciplinary connections between mathematics and art and mathematics and nature. In particular, it is suggested to consult:
- the *Contexts of Meaning* "Personal Numbers" and "Mathematics and Art," in which insights are offered into motivating situations in which the *golden number* and proportionality are implicated in the human body and in artistic contexts;
- the *Teaching Practices* "Mathematics and Art in the First Cycle" and "Mathematics and Art in the Second Cycle," in which teaching proposals related to the *golden number* and divine proportions in works of art, among others, are collected.

Among the publications outside the MaMa platform, we recommend the educational notebook "In Art... Math!" from the *Praticamente* series in which an educational experience lived in elementary school related to the discovery of the Fibonacci succession, the golden number and its applications in the art world is recounted.

A cartoon by Leonardo Pisano, known as Fibonacci, can be found in the collection "Mathematicians in Cartoons," under the heading Fibonacci (13th cent.), where his famous succession suitable for elementary and middle school is also proposed in a humorous way.

To learn more about the Fibonacci succession, you can view the entertaining video "The Fibonacci Succession" from the series "Mathing Ciak!", which consists of a collection of videos for educational use aimed at elementary and middle school.

"A Golden Figure" is the title of a story, nursery rhyme and song designed for elementary school related to the "A World Figures" project. A project curated by the Center for Mathematics Didactics of the Department of Formation Learning / Alta scuola pedagogica of SUPSI in Locarno together with RSI KIDS.

The classic Disney short film "Donald Duck in the World of Mathemagic" from 1959 should also be mentioned on this theme. Among the various contents featured, some are related to the connections between the golden number, art and nature. To learn more about the topics covered in the film, you can download from the "Matematicando" website the teaching sheets "Donald Duck's Mathemagics" for 6-10 year olds and "Donald Duck's Mathemagics" for 11-14 year olds.

Finally, the collection "Mathematics and Nature," produced by elementary school teachers from the Ticino area, presents numerous teaching ideas that leverage interdisciplinary connections between mathematics and nature and soon to be available online on the "Matematicando" platform.

The following texts, suitable for the last years of elementary school and middle school, can be used for teaching purposes to cover the topic of Fibonacci succession and the golden number in the classroom:
- Cerasoli, A. (2010). *I count*. Feltrinelli Kids.
- Cerasoli, A. (2011). *The magnificent ten*. Editorial Science.
- Enzensberger, H. (1997). *The magician of numbers*. Einaudi Ragazzi.

- Feniello, A. (2014). *The child who invented zero*. Laterza publishers.

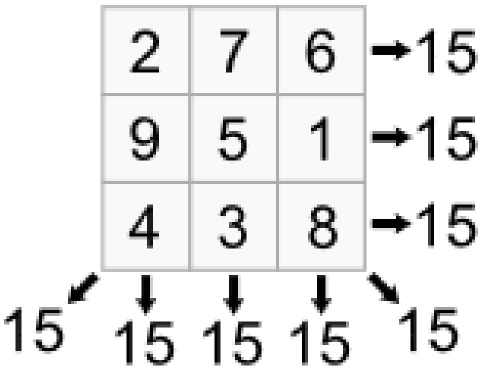
Also, an interesting comic strip suitable for older children is the following:
Flandoli, C. (2020). *Leonardo's book*. Comics&Science.

For further study aimed at adults, the following references are recommended:

- AA.VV. (2012). *The golden section*. RBA.
- D'Amore, B. (2007). *Mathematics everywhere*. Pythagoras.
- D'Amore, B. (2015). *Art and mathematics*. Daedalus editions.
- D'Amore B., & Sbaragli S. (2017). *Mathematics and its history: from the origins to the Middle Ages.* Daedalus editions.
- Livio, M. (2005). *The golden section*. Rizzoli.
- Maor, E. & Jost, E. (2017). *The art of geometry*. Code Editions.

MAGIC SQUARE

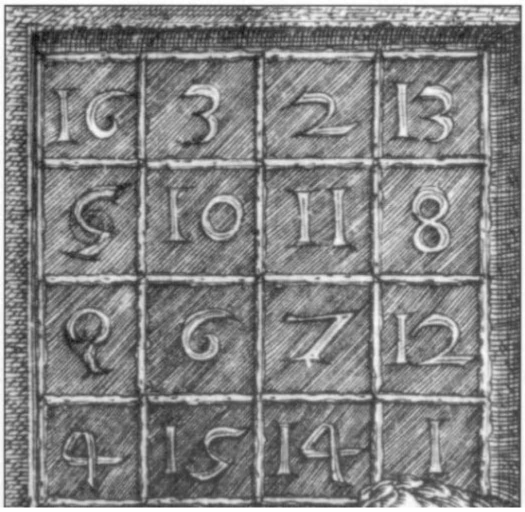
What is it?A magic square is a square table of positive integers (1,2,3,4,5,6, etc.) constructed so that the sum of the numbers on each row, on each column and in both diagonals always gives the same number called the *magic constant.* The number of columns or rows is called the *order of* the square.



Why?Man has always been attracted to mathematical games and the puzzles they involve. In fact, the magic square, in addition to being used as a puzzle, is used by various civilizations--Arabs and Greeks especially--in certain applications of mathematics, such as combinatorial calculus, particularly to discover the total number of possible combinations depending on the order.

Fun fact: An ancient Chinese legend dating back to about 2000 B.C. tells of a fisherman who found along the banks of the Lo River a turtle with strange geometric markings on its shell. The emperor's mathematicians discovered that it was a square of numbers with constant sum 15 on each row, column and diagonal. The Shu, as it was called, became one of China's sacred symbols.

One of the most famous magical squares is by Albrecht Dürer (painter and surveyor, 1471-1528) and appears in his etching entitled *Melancholia I.* It was probably the first to appear in European art and its magic constant is 34. The constant is also valid when the four corner tiles, the four center tiles, the four quadrant tiles and several other combinations are added together individually. In addition, the number 1514 (date of the work) and 34 (Dürer's age when he made the work) appear.



Teaching links: *Combinatorial calculus*.

The magic square is a particular example of the *Latin square*, "square checkerboard of side *n* with a symbol on each square so that each appears once and only once in each row and column." Another example of a magic square is the famous Sudoku.

MAGIC SQUARE OF ORDER 3 $\rightarrow\ n = 9$

A procedure for making up the magic square 3×3 (with numbers from 1 to 9) is as follows:

1. Arrange the numbers on the grid in ascending order starting from the top left box.
2. Move the numbers one box by rotating them clockwise around the 5 that is in the middle box.
3. Swap places the numbers at the ends of each of the diagonals, that is, the 4 with the 6 and the 2 with the 8.

The sum of all the numbers in the square is:

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45.$$

In more general terms, the sum of n consecutive numbers is given by $[(n+1) \times n]/2$.

In this case $n = 9$ therefore, applying the previous formula, we get just that $45 = (10 \times 9)/2$.
In fact, taking the two numbers in the outermost frame at the extremes and gradually the innermost ones, we always get the same value, viz. 10, in fact by adding the four pairs of numbers $1 + 9$, $2 + 8$, $3 + 7$ e $4 + 6$ to which is added 5, the central and only number left unpaired. Once the sum of the numbers is known and we want to arrange them in three rows (or columns) so that the sum is the same in each row, it is immediate to deduce that the constant sum (called the *magic constant*) must be $45/3 = 15$.

In general, the constant sum for a square of order n is equal to $\mathrm{n} \times (\mathrm{n}^2 + 1)/2$, a formula that can be derived by knowing that in a square of order n the numbers from $1 \mathrm{to} n^2$, which then sum to $\mathrm{n}^2(\mathrm{n}^2 + 1)/2$ and then dividing that sum by the order of the square (that is, by n).

Then there is a rule that says that the constant sum in a magic square of odd order is obtained by multiplying the number in the center by the size of the square. In our case the square has size 3 so, knowing the sum (15), we can derive that the number in the center must be 5 ($15/3 = 5$). Noting the number to be placed in the center of the square, everything is much simpler.
Wondering how many magic squares of order 3 or higher there are is a combinatorial calculus problem. The answer, however, is not immediate and was first calculated by Bernard Frénicle de Bessy (1605-1665), a French mathematician and friend of Descartes who, in 1663 calculated:

- The number of magic squares of order 3 is8, with constant sum 15, on rows, columns and diagonals.
- The number of magic squares of order 4 is 880, with constant sum 34, on rows, columns and diagonals.
- Only thanks to the computer was it possible to extend the result, in 1973, to higher orders: magic squares of order 5 are 275,305,224.
- The precise number of magic squares of order 6 is not known, although many mathematicians have engaged in its determination. According to some investigations, their number is in the range of. 1.7754×10^9.
- However, the more general problem of finding the rule for determining the number of magic squares of order n remains unsolved. This confirms that it is not easy to find a mathematical rule for all the quantitative problems we face!

Other external resources:
(English)https://en.wikipedia.org/wiki/Magic_square
Exhaustive Wikipedia page devoted to magic squares.

TOWER OF HANOI

What it is. It is a puzzle game played with three stakes and a varying number of discs. The rules are easy:

- you have to move the whole tower from one stake to another, one disk at a time;
- a large disk cannot cover a smaller disk.

In the exhibition version, the stakes are arranged in triangles instead of in rows (classic version) and you can play with up to 8 discs.

Why. This puzzle is immediately understandable, solves quickly with just a few discs (the advice is to start with 3 discs), but gets quite complicated as the number of discs increases: the number of moves needed to solve it increases very quickly. The minimum number of moves, excluding errors, is perfectly predictable: if n is the number of discs being played with, it will take at least $2^n - 1$ moves to solve it.

For example:

- With 1 disk, it will be necessary to $2^1 - 1 = 2 - 1 = 1$ move.
- With 2 disks, it will require $2^2 - 1 = 4 - 1 = 3$ moves.
- With 3 disks, it will require $2^3 - 1 = 8 - 1 = 7$ moves.

And so on.

Put another way, each time you add a floor, you need twice as many moves plus one. This second relationship is more evident than the first as you play: each time you add a disk (below the others) you will first have to repeat the same operations as in the previous round to free it, then move the added disk, then repeat the moves in reverse to cover it.

Trivia: This game has an ancient flavor, but it was invented in 1883 by mathematician Edouard Lucas, as was the story that accompanied the toy version: "*In the beginning of time, Brahma brought to the great temple of Benares three diamond pillars and 64 gold disks, placed on one of these pillars in descending order. It is the sacred Tower of Brahma that has the temple priests busy, day and night, transferring the tower of discs from the first to the third column. They must not contravene the rules imposed by Brahma, which require that only one disc be moved at a time and that there never be a disc on top of a smaller one. When the priests have completed their work and all the discs are rearranged on the third column it will be the end of the world.*"

This would happen -it is mathematical- in well $2^{64} - 1 = 18.446.744.073.709.551.615$ moves. If they were one per second (without error!) it would be more than 5 billion centuries.

Educational links: ***Exponential***.

Hidden in the game's solution are the well-known powers of 2, recalling among other things the binary notation of numbers (in fact, there is an interpretation of the game in this key, see link below).

Teaching links: ***Mathematical successions***.

To prove (by *induction*) that the minimum number of moves to solve the game with n planes is $2^n - 1$ a result from the theory of successions is used: $\boldsymbol{\sum_{k=0}^{n} 2^k = 2^{n+1} - 1}$.

Educational links: ***Programming*****.**

The triangular arrangement of the Hanoi tower also highlights the *algorithm of* moves that can be found as a solution (in a nutshell: even-numbered discs always move in one direction, odd-numbered discs move in the opposite direction). Following this procedure, a computer can solve the problem without knowing the rules of the game, while a human could move the tower correctly while thinking about the shopping list, whistling.

Other external resources:
(In English) https://en.wikipedia.org/wiki/Tower_of_Hanoi
Exhaustive Wikipedia page on the game.

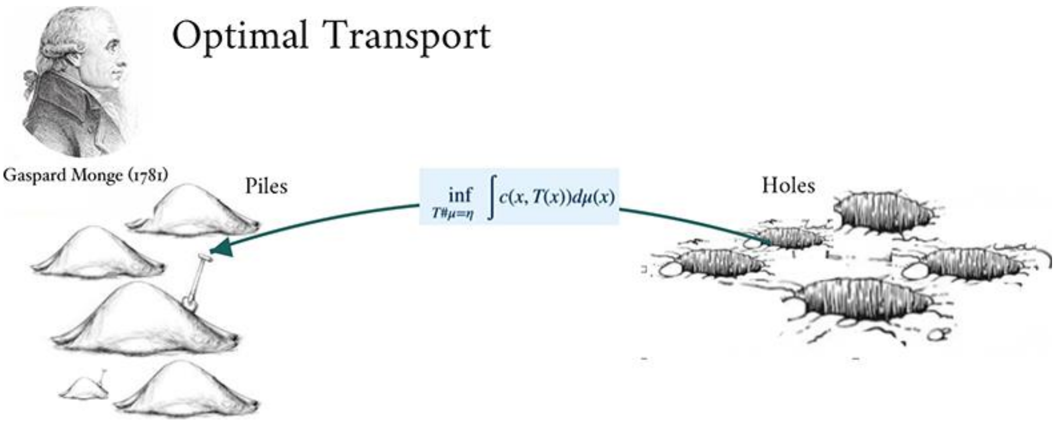
EXCELLENT STRATEGIES - THE LEAP FROGS

The material for this section on optimal strategies is a reworking from teaching materials shared by colleague Prof. Sumeetpal Singh (School of Mathematics and Applied Statistics, University of Wollongong, Australia) who received financial support from the TIBRA Foundation.

This section helps to think like a mathematician or statistician or, even more generally, like a data scientist i.e., to follow a learning process that includes the following steps: Familiarize, Visualize, Generalize, Verify. The section is organized according to these points:

1 Introduction
2 The challenge
3 Unpack the problem
4 Task1
5 Task2
6 Task3
7 Task4
8 Summary

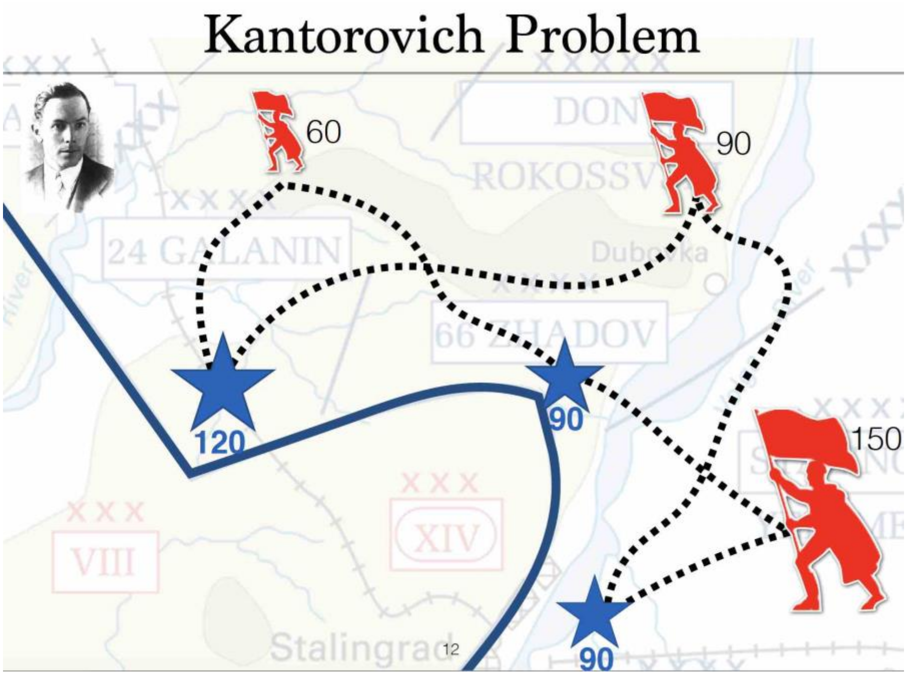
Let's start with a question: what do mathematicians do in their daily work?
Mathematicians solve real problems! Mathematicians find the best strategies.
Let us begin with an ancient problem that is still very relevant today.
Problem: Find the strategy to move soil with minimal effort from the piles to fill the holes. This is a problem called "optimal transport." Fields Medal2018 - the equivalent of the Nobel Prize in Mathematics - Prof. Alessio Figalli was awarded especially with reference to the theory he developed on this topic.



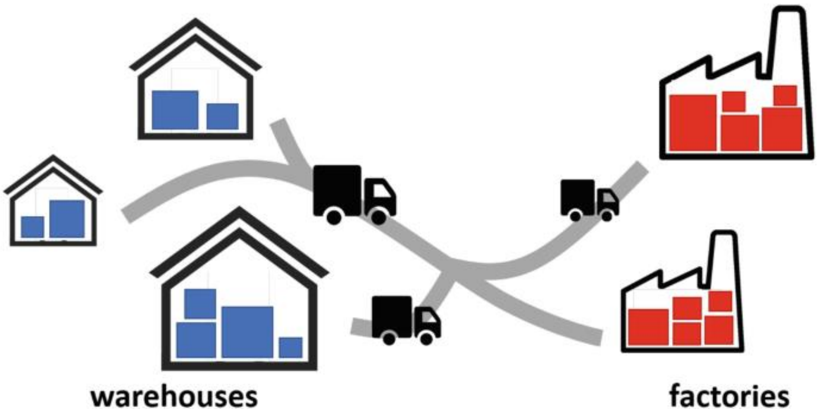
Can you think of a current version of this problem?

Here are some other examples.

Moving troops from barracks to combat posts.



Moving materials from warehouses to factories.



Computer reconstruction of how the earth has evolved to the present day.

Credits for the 3 images above: M Cuturi& J. Solomon; https://arxiv.org/pdf/2008.02995.pdf ESA and the Planck Collaboration,T.H. Jarrett & RoyaMohayaee.

Your challenge: Solve a "student version" of the Optimal Transportation problem.

Goal: Swap the position of the frogs in the least number of moves! The blue frogs from the right must move to the left, and the red frogs that were initially on the left must be moved to the right.

Starting point: Figure 1

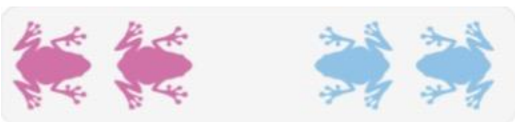


Fig. 1

Rules: Frogs can slide to empty spaces that are immediately to their right or left as in Figure 2:

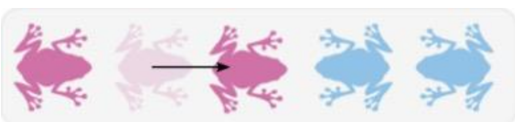


Fig. 2

A frog can jump over a frog adjacent to it as in Figure 3:


Fig. 3

For details see: https://nrich.maths.org/content/00/12/game1/frogs/index.html#/student

How do we proceed? To solve a complex problem, one possible strategy is to decompose the problem into simpler problems.

We have four tasks to do:
Task 1: Become familiar with an online game.
Task 2: Record/communicate your solution.
Task 3: Look for a model for a solution strategy.
Task 4: Verify the strategy and formula.

Task 1: Become familiar with the game. we can do this with the application created by the University of Cambridge and available online at this link: https://nrich.maths.org/1246

We start experimenting with only 2 red frogs and 2 blue frogs as in Figure 4.

Fig. 4

As an alternative, we can build frogs out of paper by following simple origami steps:
https://www.youtube.com/watch?v=FuygepwQyN8

Record the total number of movements for each attempt.
The goal is to minimize the number of moves. We soon discover that the best solution need not involve backward steps.

Task 2: Recording/communication of the solution (≈5 min).
Discuss how to record your solution so that it can be played back.

Try making the game more complex by going from 4 frogs (2 blue and 2 red) to 6 frogs (3 blue and 3 red) as in Figure 5.

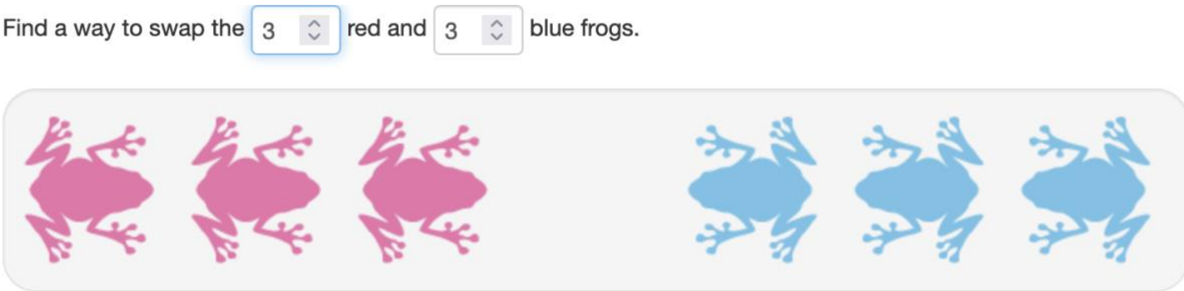


Fig. 5

Hint: How could you record movesin a board game?
We learn how to use a table to record moves (≈10 min).
A good visual can help identify a strategy to solve the problem
problem for any number of frogs.
Use the table in Figure 6 to record a solution that has the minimum moves.

R3	R2	R1		B1	B2	B3
		1				
				1		

Fig 6

Task 3: Find the optimal strategy and formula (≈12 min.)
Using the filled table:
Find the formula for the number of moves. Find the strategy for moving the frogs.

List the moves by indicating the color of the frog that moves:
R, B, B, R, RB, B, B
R, R, B, B, R

Count the moves made:
1+ 2 + 3
+ 3

+ 3+2+1
Do you see a pattern?
Can you find the number of moves for 4 frogs, or 5 frogs, or, more generally, for n frogs?

List the moves in an alternative way: this way we not only indicate the color of the frog that is moving but also whether it is the frog associated with the number 1, 2 or 3 (the numbers refer to the position that frog has at the beginning of the game, it is as if the frogs have attached a tag that shows their number which is written in red or blue depending on their color). This is how the same moves as before can be listed in an alternative way with more information content:

1,
1, 2,
1, 2, 3
1, 2, 3
1, 2, 3,
2, 3,
3

Task 4: Check the strategy and formula.
Now that you have found a formula for n frogs, verify it:
You have solved the case of the 2 frogs in 8 moves. Does the formula agree?
Play the game for multiple frogs and implement your strategy. Verify that it gives you the least number of moves. That is, verify that the minimum number of moves is correctly predicted by your formula.

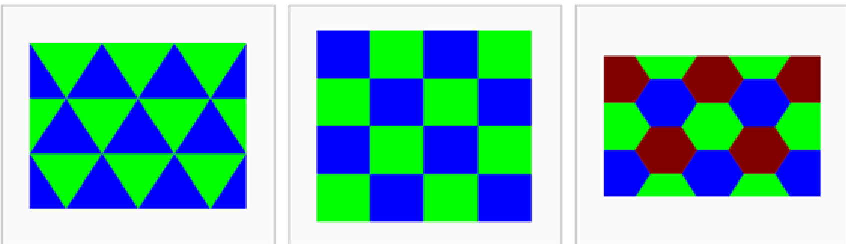
Well done, if you have come this far you have followed the typical steps of scientific research:
1 - I observe a situation and try to understand the problem I need to solve
2 - I describe it in first visual and then mathematical terms, for example with a table and looking for regularities, patterns that help me understand the situation better
2 - I try to explain the situation and a possible solution using mathematical formalism
3 - I try to figure out whether my formalism, in this case the formula, is verified in general

By following these simple steps, one after another, Science has taken us all the way to the moon ... and beyond!

TESSELATION

What it is. A tessellation is the periodic repetition of the same figure covering a plane without overlapping or holes. The figures used to cover the surface, tessellations, are often polygons, regular or not, but they can also have curved sides or be without any vertices.

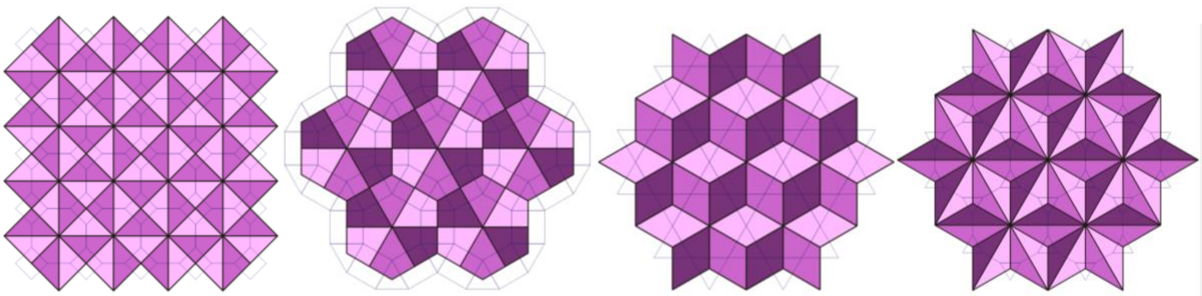
Why? Some figures, e.g. triangle, square, rectangle, hexagon, etc., allow us to make a complete tessellation, while others do not. How come? And how can the result be predicted? Mathematics helps us: in general, it is enough to add up the amplitudes of the angles formed at any of the meeting points of the tessellations to figure it out. If the result of adding the amplitudes of these angles is different from 360°



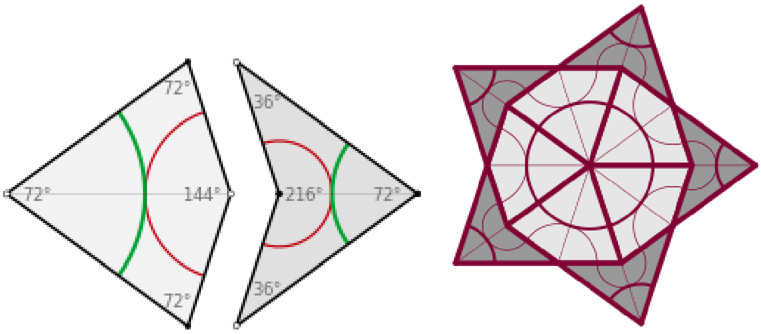
Tasselli triangolari Tasselli quadrati Tasselli esagonali

complete tessellation will not be possible, otherwise yes. If we then impose for the whole tessellation the use of only one regular polygon, viz.

with equal sides and angles, we have only 3 possible configurations. In fact, in this case the measure of the angles of the dowel will have to be an integer divisor of 360° and therefore only the equilateral triangle (60°), the square (90°) and the regular hexagon (120°):



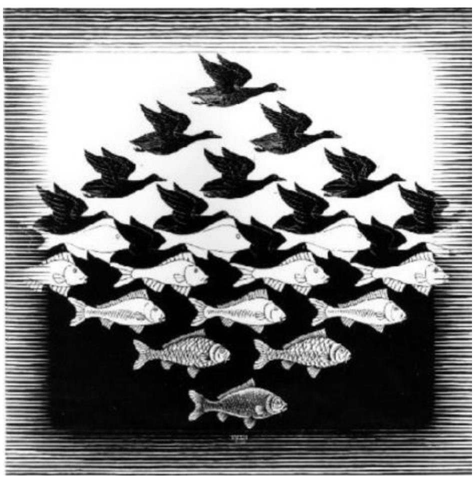
Fun fact: The tessellations shown above are called *regular tessellations* because they each use only one geometric shape. In 1974 Roger Penrose and Robert Ammann discovered various *irregular tessellations*, that is, making use of multiple geometric figures in a single tessellation, including the kite and the dart.



In nature there are *infinite* tessellations, that is, whose basic geometric shape is continuously repeated. An example is the hexagon in the bee hive:

Dutch artist Maurits Cornelis Escher is famous for his tessellations that often depict animals such as fish, birds, and horses.

M.C. Escher, *SkyandWater I*, 1938



Educational links: ***Geometry***.

Covering the plane or space with repeating regular figures is a classic problem in geometry, even before it was an artistic one. It involves a whole range of geometric concepts such as reflections, rotations, translations and, not least, the *golden section*.
More generally, objects such as the tangram/stomachion, the magic square, the Tower of Hanoi, and the tessellated plane put systems with very precise rules into game form, allowing a whole series of outcomes that are often invisible at the outset to be deduced and thus predicted with certainty. Although this does not make them trivial or easy to solve (the case of Stomachion is emblematic). Quite the opposite of games based on chance, such as games of chance.

Educational links: ***Philosophy***.

What Aristotle began more than 2,000 years ago, a team of 30 undergraduate students at the Massachusetts Institute of Technology is continuing now. They are starting with a recent mathematical result that has breathed new life into the age-old hunt for geometric shapes that can tessellate, that is, perfectly fill, three-dimensional space. "It is exciting to know that some of the greatest minds of all time have been working on this topic.

Other external resources:
(In English) https://en.wikipedia.org/wiki/Tessellation
Exhaustive Wikipedia page on tessellations.
(In English) https://en.wikipedia.org/wiki/Penrose_tiling
Wikipedia page on Penrose tessellation.

PLANTING (FEEDING) GAMES

Planting games are a family of board games widespread in much of the world (especially in Africa, the Middle East, Southeast Asia, and Central America). The similarity of many aspects of the game to agricultural activity and the simplicity of the board and pieces, the large number of variations, and their spread throughout the world suggest an extremely ancient origin; according to some, perhaps, close to the very origins of civilization.

These are ancient games that are precursors to backgamon. Play is a universal tool that helps to learn more about oneself and others. Play is pastime, recreation and at the same time a vehicle that allows, almost without realizing it, to appropriate useful skills in the development of each individual.

Mancala is a very ancient African game that is part of the family of planting games.
In one of the most popular versions, the mancala is a board that has two parallel rows of six holes, one row for each player. Four tokens, the *seeds,* are placed in each hole. On either side two receptacles, the mancala, hold the earned seeds. They can be beans, seeds, berries or grains, stones or as we did, shells.
The board represents heaven and earth, and in some cultures the movement of the tokens simulates the acts of planting and harvesting.

Rules:
In turn each player takes all the tokens contained in one of the six holes in his row and, proceeding counterclockwise, "plants" them in the holes that lie in succession after the one from which he took them, one per hole including of course his own mancala.
If, during play, either player arrives with his last token in an empty hole or one containing more than two marbles, he takes none. If, on the other hand, the hole contains 1 or 2 checkers, excluding the one he laid down, he is entitled to take all the checkers in the hole, including his own.
The game ends when either player no longer has enough checkers to move. In this case, the opponent captures the remaining checkers and the player who has captured the most checkers wins. However, to prevent such a situation from occurring too early in the game, it is forbidden to completely empty the opponent's ranks, thus preventing him from making a counter move, unless all possible moves block, in any case, the opponent. The player who does not comply with this last rule is punished: his opponent captures all the pawns left in the game.

NUMB3D BY NUMB3RS!

DICE

QUINCONCE OR GALTON'S MACHINE

The Galton machine is named after Sir Francis Galton, an English mathematician and statistician who developed it in the late 19th century to explain some statistical concepts. Sir Francis Galton was a cousin of Charles Darwin and was greatly influenced by Darwin's work on evolution and natural selection.

How to play?

STEP 1: Drop a ball first and try to guess which bottom box it will end up in. It is quite difficult to predict!
If a child guesses exactly the box, he gets 3 points.
If he/she misses by one square (left or right), the child gets 2 points
If he misses by 2 squares he receives only 1 point.
Otherwise zero points.
The person who scores the most points wins.

PHASE 2:
Now drop a handful of balls and try to guess how many will end up in the most lateral boxes (far right or far left).
If the child says 10 balls will fall in the far right box and no balls end up in that box, he loses 10 points.
If the child says 10 and 1 ball ends up on the far right, he loses 9 points (10-1). And so on.

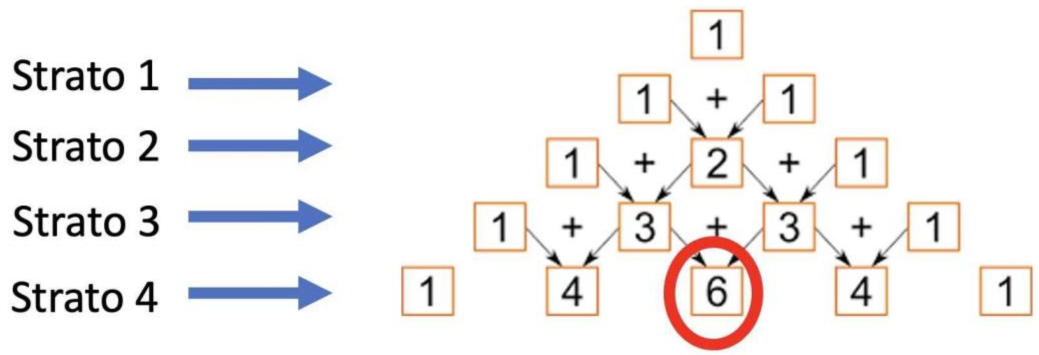
STEP 3:
Now drop all the balls.
What do you think will happen?
Can we predict where they will end up?

What have we learned?
At the top of Galton's machine, the balls fall haphazardly and chaotically but, "as if by magic," at the base they are funneled into different boxes, ALWAYS forming the outline of a "bell" known as a **Gaussian or normal distribution**.
How can this regularity be justified?
On each peg the individual ball can "decide" to bounce left or right.
If the ball "wanted" to go to the last box on the right, it would have to "decide" to always bounce to the right.
In other words, **there is only one path** that takes the ball to the last square on the right.

The same applies to the last box on the left.
Instead, there are **many more paths** that take the ball to the center squares.
For example, a bounce to the right followed by one to the left and so on bring the ball to the center square.

But also: two bounces to the right and two to the left and so on bring the ball to the center square. In general, it is sufficient for the number of steps/rebounds taken on the right to be equal to the number of steps/rebounds taken on the left for the ball to end up exactly in the center square.

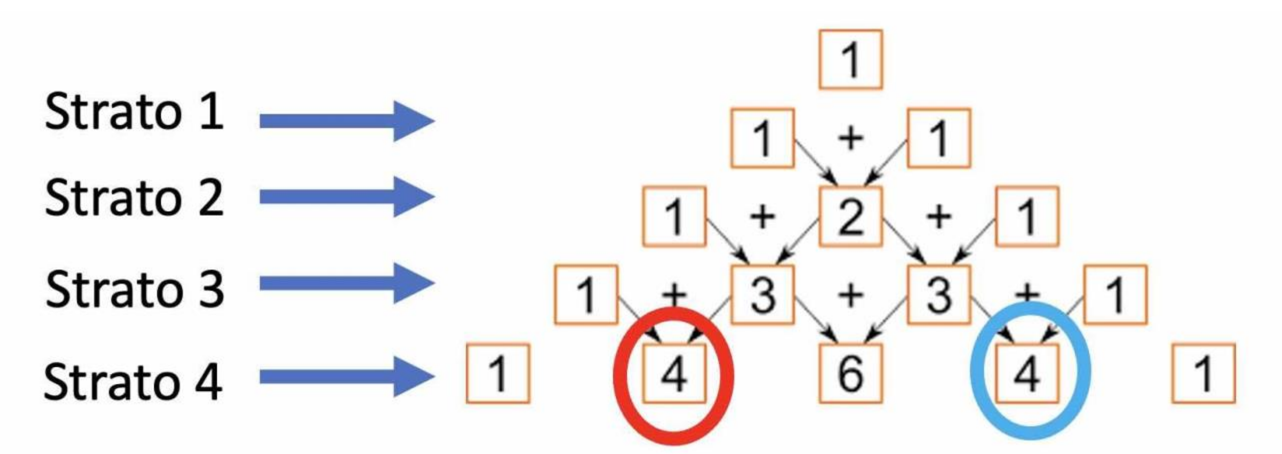
If there were only 4 layers of pegs, the number of paths leading to exactly the center square would be 6, as shown in the last row of the Tartaglia triangle below. Can you find all the possible paths leading to the center square?



In Tartaglia's triangle, each number is the sum of the numbers preceding it, as in the image.

Exercise: complete the triangle with more layers.

Still with 4 layers, the number of paths leading to exactly the second-to-last box on the left would be 4, as shown in the last row of the Tartaglia triangle below.



Can you find all these paths?
And what is the number of paths leading to the second-to-last box on the right?
There are interesting symmetries!

If instead of 4 there were 6 layers of pegs at the top of this post, the number of paths leading exactly to the center would be 20 as shown in the last line of Tartaglia's triangle shown below.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

Tartaglia's triangle is also important in calculating the coefficients in the development of the n-th power of the binomial $(a+b)^n$.

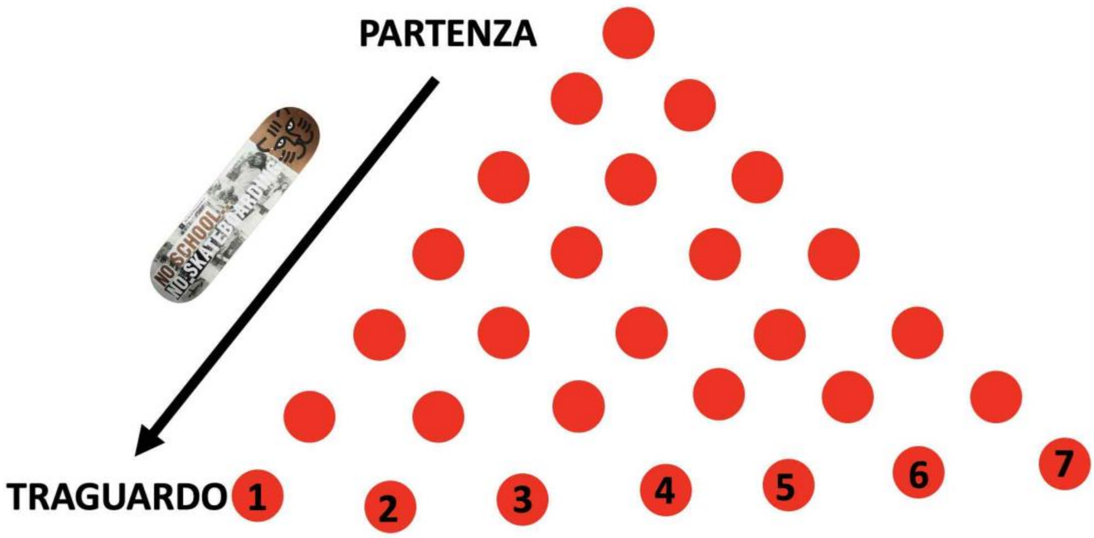
The Gaussian distribution is symmetrical with respect to the mean, which is also the mode and median, and this symmetry is "similar" to what we observe in Tartaglia's triangle.
It is the Central Limit Theorem that underlies the regularity we observe in the boxes of this station in which the balls are channeled in an orderly mode, an order that contrasts with the randomness observed at the top of the station. It is also because of this regularity, dictated by the Central Limit Theorem, that statisticians are able to make predictions based on large samples.

What do we learn from this game with reference to everyday life?

- Each ball can be considered as a person
- When people make binary decisions (with only 2 possible options, such as to vote or not to vote, to go to school or not to go to school, to travel or not to travel) they behave like a ball that "decides" to go left or right.
- If people's decisions are independent of each other, we can study the result of many decisions made by many people using the Gaussian curve.
- Carl Friedrich Gauss was a German mathematician, physicist and astronomer who lived from 1777 to 1855. He is considered one of the greatest mathematicians of all time.

Galton Skate-Board on Paper

Print (or draw) this image:



GALTON-SKATE-BOARD !

How do you play with the Galton-Skate-Board?

- Each child has an object that represents his or her skateboard.
- Each child has a coin that is tossed 6 times.
- The skateboard starts at the "start" and ends at the "finish" line.
- Each time the coin toss is HEAD, the skateboard slides one step toward the red circle on the right.
- Each time the coin toss is CROSS, the skateboard slides one step toward the left red circle.
- In what final position (1, 2,… or 7) will most skateboards end up?
- NOTE: The more children play, the less uncertain the outcome will be!
- Each child can play several times while keeping track, each time, of the final position.
- You can replace coin tosses (with heads and tails) with stones, black and white.
- You have a bag with an equal number of black and white stones and in turn the children select 6 stones: if the stone is white they slide their skateboard to the left red circle, if it is black they slide their skateboard to the right red circle.
- Once a child is finished, his stones are put back into the bag and the next child begins.
- Let us reflect together: what happens if the number of white stones is greater than the number of black ones?

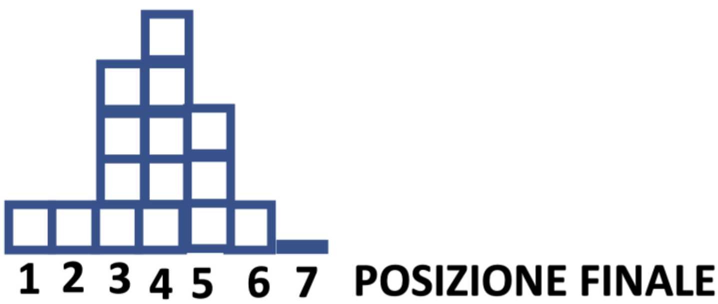
Enter each child's final skateboard position in each play in a grid similar to the one below.

	GAME 1	GAME 2	GAME 3
BABY 1			
BABY 2			
BABY 3			
BABY 4			
BABY 5			

EXAMPLE of the outcome of the Galton-Skate-Board game

	GAME 1	GAME 2	GAME 3
BABY 1	Final position 5	Final position 4	Final position 3
BABY 2	Final position 4	Final position 3	Final position 5
BABY 3	Final position 4	Final position 1	Final position 4
BABY 4	Final position 3	Final position 4	Final position 5
BABY 5	Final position 6	Final position 3	Final position 2

Posizione finale 1: raggiunta 1 volta
Posizione finale 2: raggiunta 1 volta
Posizione finale 3: raggiunta 4 volte
Posizione finale 4: raggiunta 5 volte
Posizione finale 5: raggiunta 3 volte
Posizione finale 6: raggiunta 1 volta
Posizione finale 7: raggiunta 0 volte



The image you see is called a histogram and is a graphical representation of the data.
It consists of a series of bars of equal width. The height of each bar represents the frequency or number of observations of each outcome in the game.

EXPLANATION

- Most skateboards will end up in position 4

- Very few will end up in positions 1 and 7.
- Why?
- There is only one path leading to the final position 1: C, C, C, C**, C, C**
- There is only one path leading to the final position 7: **T, T, T, T, T**
- There are many paths leading to the final middle position 4, for example:
  C, T, C, T, C, T oppure T, T, T, C, C or C, C, C, T, T, T
- In general, each path with **3 Cs** and **3 Ts** leads to the final position 4

REAL APPLICATIONS

Galton's machine illustrates the Central Limit Theorem, a theorem that is CENTRAL to predictions that become more reliable as the "sample size" becomes larger and larger (in the LIMIT).

<u>**EXAMPLE 1) Children's height**</u>

- Measure the height of 20 children and calculate the average height.
- We call this quantity the **population average.**
- Each child now takes a piece of paper and writes his or her name.
- The sheets are placed in a box and 10 names are selected at random (without looking). The average height of the selected children (sample) is calculated. We call this average the **sample average.**
- The central limit theorem tells us that the sample mean should be very close to the population mean.
- If fewer children are selected in the sample, say 5, the two averages (population and sample) should still be close, but probably less close.
- If only one child is selected, it is difficult to say anything about his or her height.
- This is similar to what happens when we throw a single ball into Galton's machine: it is difficult to predict where it will end up.
- If we have a handful of balls (equivalent to a sample) then it is easier to make predictions.

<u>**EXAMPLE 2) Regional average salary.**</u>

- Salary of people in a region: it is too time-consuming to ask each person in a region for his or her salary and then calculate the average salary for the region.
- It is much easier to take a random (representative) sample of people in the region and calculate the average salary of the sample.
- The central limit theorem tells us that the average wage in the region is very close to the average wage in the sample. And it gets closer and closer to the true value the larger the number of people in the sample, as long as the sample is representative.

LOT-POT

What it is. The lotto game consists of drawing without re-entry from an urn, some balls all numbered differently. The winner - when there is one! - is the person who will have been able to predict part or all of the numbered balls drawn. In Switzerland, the lotto game was introduced in 1970, and the first million-dollar payout came in 1979. In the beginning it was played with 40 numbers then changed to 42, in 1986 to 45 and since 2013 back to 42, of which 6 numbers are drawn. The probability of guessing 6 numbers out of 42 is 1 in 5,245,786.

Why? When playing Lot (Lotto) we are tempted to bet on the numbers that have not come up for the longest time, the so-called "laggard" numbers. Unfortunately, however, it is only chance that decides: there are no tricks or methods, as every number has the same chance of coming out. The probability of a number coming out corresponds to 1/42, just over 2 percent. A number may lag hundreds of draws but its probability of coming out always remains 1/42, not increasing, let alone decreasing. This type of game has no memory; yet, we are led to believe otherwise.

But is lotto a **fair** game? A *game is* said to be *fair* if it matches a prize that depends on the odds of winning. In tossing a coin, the probability of guessing whether heads or tails will come out is 1 in 2. The game is fair if betting one franc can win 2. In all other cases the game is unbalanced, usually in favor of the bank. Lotto is not a fair game; the banker enjoys an advantage secured by the high number of bets and the unfairness of the odds paid as a premium compared to the probability of winning. Therefore, although the player sometimes wins-even substantial prizes-he is always a loser in the long run.

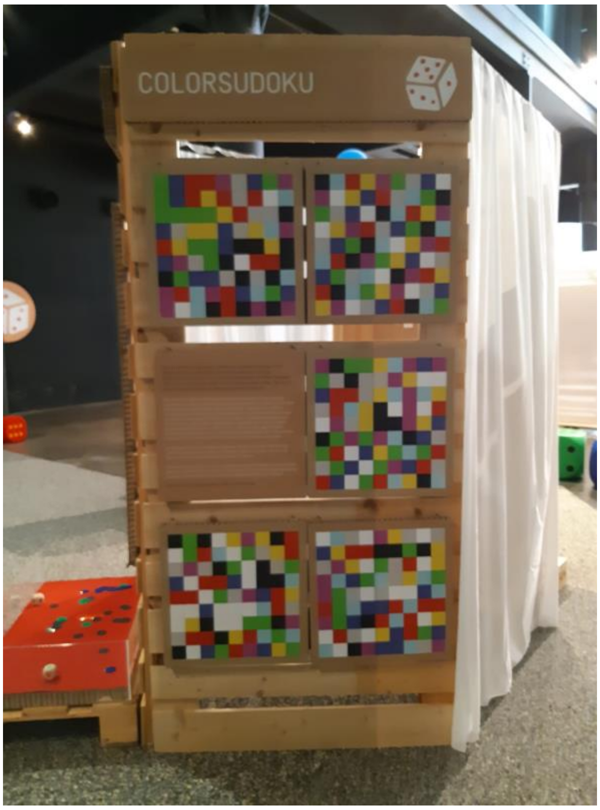
What is the probability of guessing 3 out of 42 numbers? Suppose we choose: 2, 11, 29. These can be drawn in 6 different ways, e.g. (6=3!=3x2x1; n!=nx(n-1)x(n-2)x...x1): 2,11,29 - 2,29,11 -29,2,11 - 29,11,2 - 11,29,2 - 11,2,29. How many triplets are possible with 42 numbers? You have to combine 42objects 3 by 3, in mathematical terms it is equivalent to calculating 42!/(3!x(42-3)!) i.e.

(42x41x40)/6=11,480. So the probability of guessing a trio with 42 numbers is 1 in 11,480. Low? With the same calculations we find that the probability of guessing 6 numbers out of 42 is 1 in 5,245,786. It has been calculated that it is more likely that asteroid 99942 Apophis in 2036 will hit the Earth-the probability is 1 in 40,000). But there are so many players that sometimes someone actually guesses them.

Trivia: Twelve thousand charitable projects in the areas of culture, sports, environment and social works are supported each year with *Swisslos* proceeds: every day one million Swiss francs of *Swisslos* net profit flows into cantonal funds. Since its creation, *Swisslos* has invested about five billion Swiss francs in charitable and public benefit projects what makes *Swisslos* the most important Swiss promoter in the fields of culture and sports.
The Swiss lotto, since its existence, has created eight hundred millionaires.

Other external resources:
http://old.sis-statistica.org/magazine/spip.php?article172
What does statistics have to do with lotto?
http://www.festadellamatematica.it/doc2013/Antonelli-lottologia-CC.pdf
Consideration of lottology

COLOR SUDOKU



All the colored panels in this post were generated by randomly placing the nine colors in the 9x9 grid except for one of the panels (the one in the upper right corner) which is deterministic (one color per row, per column, per diagonal and in the 3x3 sub-squares). Recall the magic square. Recall that causality can create patterns that we do not expect. The visitor is asked: what is the "different" panel. Why do you think this one is different?

This station can be accompanied by the "magic" game of guessing which among two sequences of heads and crosses in a thought experiment and an actual coin toss 10 times is the "made-up" one.

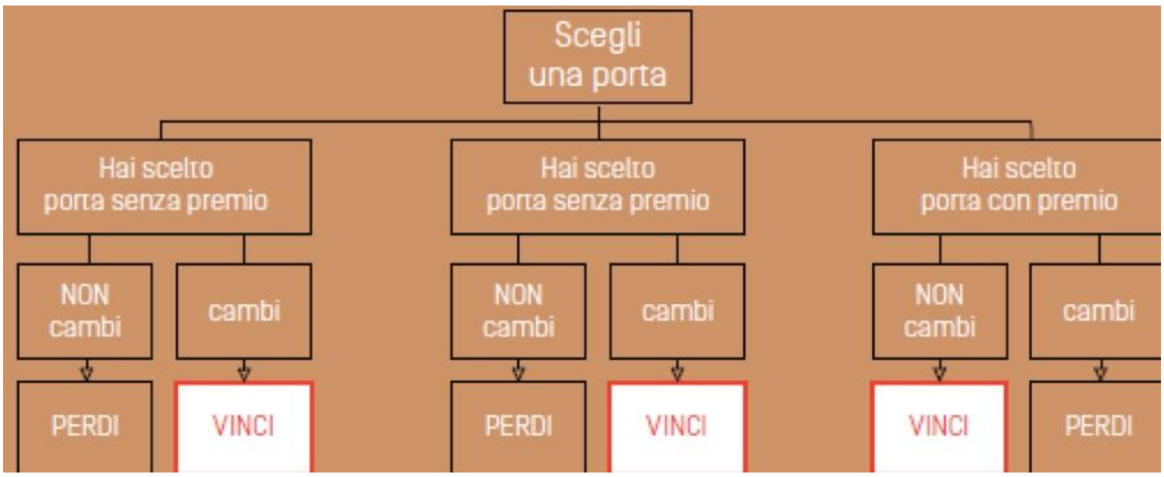
Have the visitor write down the two sequences on a sheet of paper in an order chosen by the visitor, and then the animator guesses which sequence is the actual sequence (i.e., obtained from the actual coin toss) and which was made up. In the actual toss, there are more likely to be sequences of 3 or more equal sides of the coin.

MAKE YOUR CHOICE

What it is. This paradox is inspired by the 1960s American TV quiz show *Let's make a deal*, hosted by host Monty Hall. The game consists of three doors, behind each of which is a prize or a *non-premium* the prize is hidden behind a single door. The probability of the prize being behind a given door is the same for all doors and is 1 in 3.
The player chooses one of the doors and tells everyone. The handler knows what lies behind each door and must open one of the doors not selected by the player and absolutely one without a prize: if the player has chosen a losing door, the handler will open the other losing door; if, on the other hand, the player has chosen the winning door, the handler will open one of the two remaining doors at will.

After opening the first door, The Host gives the player the option of discovering what lies behind the door he or she initially chose or to change. What is the best strategy for the player? Keep the first choice or change?

Why? It would seem a paradox but it is better to change the initial choice: doing so doubles the probability of winning. How is this possible?



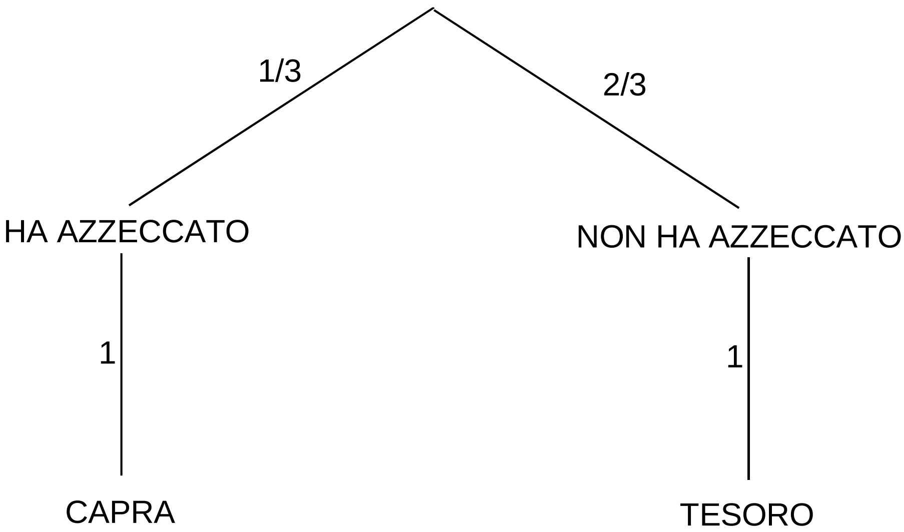
At first there are three possible scenarios, each having probability 1/3:
The player chooses the first non-winning door (number 1), The host shows the other non-winning door, (number 2): by changing the player wins. By not changing he loses.
The player chooses the second non-winning goal (number 2). The host chooses the other non-winning door (number 1): by changing, the player wins. By not changing, he loses.
The player chooses the winning door. The handler shows one of the two losing doors. By changing, the player loses. By not changing, he wins.

In the first two scenarios the player wins only by changing; in the third scenario the changing player does not win: the "changing" strategy leads to victory two out of three times (2/3), while the "not changing" strategy leads to victory in only one out of three cases (1/3).

Fact: The resolution of this problem is so counter-intuitive that several academics did not recognize it until it was explained to them in detail. A simple way to understand it is to increase the number of doors: let's put this time 4 doors, 3 losers and one winner. The player, as before, chooses one door, the host, however, shows two - losers, as before - and later asks the player if he wants to change or keep his initial choice. Thus there will be 3 out of 4 eventualities in which the player wins only if he changes and 1 out of 4 in which he wins only if he does not change.
The result is valid and, indeed, is even clearer the more you increase the number of ports.

Another way of representing the solution to Monty Hall's problem.
(with 3 doors, 2 goats and 1 treasure)

If the player decides to stay on the first choice, the probability of getting it right is 1/3 (obvious).
If he decides to reject his initial choice, the sequel can be represented with a simple tree:



$$P(TESORO) = 2/3 \cdot 1 = 2/3 = 2/3 > 1/3$$

ROULETTE

What it is. Roulette is a game of chance played on a gambling table on which there is a disc divided into 37 boxes/sectors numbered from 0 to 36 (00 to 36 in the case of American roulette), colored red and black, the box with the number 0 is green, and a betting mat.
The game is to guess which numbered sector the ball will stop on.

There are different types of winning possibilities, and these are indicated on the betting mat: even/odd, red/black, exact number, pairs/squares of numbers, first third, second third, third third third, etc. to which correspond individually a prize proportional to the risk of the bet.

Why? What is the probability of winning at roulette by betting on a specific number? 1 in 37 (1 in 38 for American roulette). And how much do you win?

We denote PO the "stake played," PV the "probability of winning," and FA the "factor multiplying the stake."
But what stakes are favorable to the player? To figure this out, just calculate the product between the probability of winning (PV) and the factor (FA) that multiplies the stake.

$$PV \times FA$$

<u>**Case 1**</u>: The player bets on a number, if he guesses correctly, he receives 36 times the OP, therefore

$$PV = \frac{1}{37} \qquad FA = 36 \qquad PV \times FA = \frac{1}{37} \times 36 = \frac{36}{37} < 1$$

So the game is favorable to the casino.

<u>**Case 2**</u>: Player bets on 4 numbers, receives 8 times the OP, therefore.

$$PV = \frac{4}{37} \qquad FA = 8 \qquad PV \times FA = \frac{4}{37} \times 8 = \frac{32}{37} < 1$$

Again, the game is favorable to the casino.

<u>**Case 3**</u>: Player bets on one color (18 numbers are red, 18 numbers are black and 0 is green), receives 2 times the EO.

$$PV = \frac{18}{37} \qquad FA = 2 \qquad PV \times FA = \frac{18}{37} \times 2 = \frac{36}{37} < 1$$

Again, the game is favorable to the casino.

The game is therefore always more or less favorable to the casino. The game would be fair if the product between the probability of winning (PV) and the factor (FA) multiplying the stakes were equal to 1; the further this number is from 1, the less fair the game is. The more it is less than 1, the more the game is in favor of the casino.

Trivia: Where did the game of roulette come from? There are many stories about its origins. The earliest examples of roulette date as far back as the 10th century AD. In its original form, it is thought to have been invented by a Chinese monk in the 14th century and imported to Europe by a Jesuit missionary. However, it is commonly attributed to French mathematician Blaise Pascal (1632 - 1662), one of the greatest probability scholars.

LUDOPATIE

Material related to ludopathies is taken from Azienda Ligure Sanitaria - Regione Liguria - ALISA, CNR

The earliest evidence of gambling dates back more than 6,000 years and has been found in Egypt, China, Babylon, and India. Today, gambling has taken on significant dimensions and a strong commercial drive, which can also be perceived through advertising. However, compared to the past, it has undergone quite a few changes: it is often machine-mediated, takes place very quickly, and increasingly targets the individual. Data from the ESPAD sample study indicate that 40 percent of Italian students aged 15-19 have gambled during the year. Among them, 11 percent are at risk of developing an addiction and about 8 percent have already engaged in problem behavior. Pathological gambling is recognized as an addiction by international nosographic systems (DSM V - gambling disorder). It determines psychological and cognitive alterations that find a scientific explanation in the study of the neurobiological characteristics of the brain and its functions. Science notes that gambling is triggered by stimuli and impulses of an exogenous nature (visual, auditory, tactile, olfactory and gustatory) and of an endogenous nature (mnestic evocations and/or visceral sensations). The workshop aims to inform about the difference between gaming and gambling, making people aware of the risks of behavioral addiction. Through the use of installations, interactive and team activities, participants will experience the mechanisms that lead the stimulus to become conditioning, test their senses and understand the related psychological and cognitive alterations.

Some interesting data on gambling can be found at the link
https://lab.gedidigital.it/finegil/2017/italia-delle-slot/

For further study, we recommend reading: DAZZARDO GAMES AND PROBABILITY (edited by P. Monari), Editori Riuniti, University Press.

What is gambling?

A bet, usually money or assets, is placed on the outcome of a future event. The bet, once placed, cannot be withdrawn. The outcome of the bet is governed by chance. The skill of the player does not matter.

Edvard Munch - At the Roulette Table in Monte Carlo

In Italy, the oldest lottery dates back to the 16th century and still bears the same name: Lotto. In Genoa, for the first time in Italy, the game of Lotto was legalized in 1576, following a long unrecognized tradition of betting on many events (outcomes of Doges' elections, marriages, sex of the unborn).
In Italy Betting on horse racing became a popular game of chance with the birth of Totip in 1948.
In Britain in the 12th and 13th centuries, betting on horse racing, now among the most popular forms of gambling, was called "the sport of kings." But in 1661 the first law banning gambling was enacted, with the aim of preventing less well-off people from getting into debt.
In France, the philosopher Blaise Pascal in the 16th century was the inventor of roulette.
In California in 1885, American Charles Fay built the first *SlotMachines.*
(Source: Arnold P The Enrucionedio of Gomblino Chartwell Books 1977).
Roger Caillois (1913 - 1978) French writer, sociologist, anthropologist, and literary critic wrote the best-known Classification of GAME (1962) in which the following definition appears: "*ALEA games whose outcome can be determined solely by fate, as in the case of the toss of a coin, in betting, roulette, lottery...In this type of games: the skill of the player is often irrelevant, chance is dominant, risk is always present.*"

Gambling: from gaming to pathology
Classification: addiction category,

Subcategory: Non-substance use disorder.
This is recurrent and persistent problem gambling behavior that leads to stress or clinically significant worsening.

GAME OF LEAPFROGS AND ZARA

Horsemanship game: there are eleven little horses numbered from 2 to 12.The numbers of the little horses correspond to the result of the sum of the throw of two regular dice (each with 6 sides and numbers from 1 to 6). The horsies advance one step each time their number comes out as the sum of the two dice rolled. The dice rolls are repeated until one of the horses reaches the finish line. Which horse is best to bet on? Which horse is most likely to win?

Why. The game highlights how it is not convenient to bet on the horses closest to the extremes i.e. with number 2 or 12 being the slowest ones. In fact, these numbers are less likely to come up in the roll of two regular dice since they can only be obtained with a few combinations.
For example, leapfrog 11 can only advance with two combinations 11= 5 + 6 and 6 + 5. While the middle numbers are much more likely to come out.
For example, 7 = 1+6, 6+1, 2+5, 5+2, 3+4 and 4+3.
In total, if you roll two dice-with values from 1 to 6-you can get 6x6=36 results (possible cases). The frequencies of the possible sums are as follows:

Horse 2: 1+1 =	1/36Horse	12: 6+6 =	1/36
Cavallino 3: 1+2, 2+1 =	2/36Horse	11: 5+6, 6+5 =	2/36
Grasshopper 4: 1+3, 3+1, 2+2 =	3/36Horse	10: 4+6, 6+4, 5+5 =	3/36
Grasshopper 5: 1+4, 4+1, 3+2, 2+3 =	4/36Horse	9: 3+6, 6+3, 4+5, 5+4 =	4/36
Cavallino 6: 1+5, 5+1, 4+2. 2+4, 3+3 =	5/36Horse	8: 2+6, 6+2, 3+5, 5+3, 4+4 =	5/36
Cavallino 7: 1+6, 6+1, 2+5, 5+2, 3+4, 4+3, =	6/36		

7 is the most likely outcome. This means that it pays to bet on horse number 7; however, there is no certainty of winning. Chance can make any of the 11 horses win.

An even more intuitive explanation can be obtained by playing the same game with 3 little horses: one black, one gray, one white. To advance them you draw 2 balls from a bag. If we get 2 black balls the black horse advances, if we draw 2white balls the white horse advances, if we get one black and one white ball the gray horse advances. The possible outcomes: 2 (colors) x 2 (balls drawn) are 4. Which horse is more likely to win?

Black pony:	black + black=	¼ (25%)
White horse:	white + white=	¼ (25%)
Gray horse:	white + black or black + white =2/4=1/2	(50%)

The favored horse is therefore the gray!
Using this game as a starting point, one can reflect on the issue of horse betting and hook into raising awareness of gambling addiction (see tab on roulette).

Trivia: Zara, from the Arabic word *zar*, dice, is a gambling game that was widespread as early as the Middle Ages and consists of the rolling of three dice on which each player in turn bets the output of a number from 3 to 18 corresponding to the sum of the three numbers indicated by the dice. Whoever guesses first wins. As early as the 1600s, however, many hardened players noticed that 10 and 11 came up more frequently than 9 and 12, but they did not understand why since these could come up in the same number of combinations. The problem was so keenly felt that the Grand Duke of Tuscany asked Galileo Galilei to study it. Galilei wrote:

> *"...even that 9 and 12 in as many manners are composed as 10 and 11, so that of equal use they must be reputed, it is seen no less that long observation has made the players esteem 10 and 11 more advantageous than 9 and 12."*
>
> (Galilei G., *Sopra le scoperte de i dadi*, probably written in the years between 1612 and 1623, shortly after Galilei's arrival in Florence)

Galileo noted that both 10 and 11 "are obtained by the same number of *triplicities"* (the values visible on the three dice). These triplicities can be of three kinds: those that *are made up of 3 equal numbers,* which can be obtained *in one way*, those *that arise from 2 equal numbers and the third different one* that *are produced in 3 ways* (ex.: triplicity: 1.1.2 can also be obtained with: 1.2.1 and with 2.1.1) and those arising from *3 all different numbers* that are *formed in 6 manners* (e.g.: 1.2.3; 1.3.2; 2.1.3; 2.3.1; 3.1.2; 3.2.1).

Thus, counting right, both 10 and 11 can be obtained in twenty-seven different ways (orderings), while 9 and 12 in only twenty-five ways despite having the same number of combinations, that is, six:

9 [6,2,1]*6 + [5,3,1]*6 + [4,3,2]*6 + [5,2,2]*3 + [4,4,1]*3 + [3,3,3]*1	= 25	orders
10 [6,3,1]*6 + [5,4,1]*6 + [5,3,2]*6 + [4,4,2]*3 + [4,3,3]*3 + [6,2,2]*3	=27	sorting
11 [6,4,1]*6 + [6,3,2]*6 + [5,4,2]*6 + [5,5,1]*3 + [5,3,3]*3 + [4,4,3]*3	=27	orders
12 [6,5,1]*6 + [6,4,2]*6 + [5,4,3]*6 + [6,3,3]*3 + [5,5,2]*3 + [4,4,4]*1	=25	orders

This gives a slight advantage to 10 and 11. The probability of 10 or 11 coming out is 12.5 percent, of 9 or 12 coming out is 11.6 percent, a small but decisive difference in percentages.

Galilei conclude la sua analisi scrivendo:

> *"…da questa tavola potrà ogn'uno che intenda il gioco, andar puntualissima- mente compassando tutti i vantaggi, per minimi che sieno, delle zare, de gl'in- contri e di qualunque altra particolar regola e termine che in esso giuoco si osserva, etc. "*

A table similar to the one created by Galileo can be found in the figure below:

DADO 1 \ DADO 2	*1*	*2*	*3*	*4*	*5*	*6*
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Combinazione	2	3	4	5	6	7
Frequenza	1	2	3	4	5	6
Tot. Casi	36	36	36	36	36	36
Probabilità	**3**	**6**	**8**	**11**	**14**	**17**

Combinazione	12	11	10	9	8
Frequenza	1	2	3	4	5
Tot. Casi	36	36	36	36	36
Probabilità	3	6	8	11	14

HAPHAZARD WALK

Exhibit available online.

What it is. The game consists of tossing a coin: each time it comes up *heads we take one* step to the right, if it comes up *tails* we take one step to the left. How far will we have moved after *n* steps from the starting point-which in our game is represented by a lamppost? Again, intuition leads us toward an often incorrect solution.

Why? If we take the case of a walk in one dimension, since I have the same probability of moving left and right, e.g. 50% probability, the average displacement from the origin after a certain number *N* steps is zero, i.e. on average, I take the same number of steps left and right. However, if we consider the average distance from the origin, things change: suppose we start from position 0, how far will I be after *N* steps? This distance varies each time we repeat the experiment, but if I repeat it many times, I can calculate the average distance from the initial position.
Statistics tells us that after *N* tosses of the coin, the distance from the origin (position 0) will be $\sqrt{N}$ (square root of N).

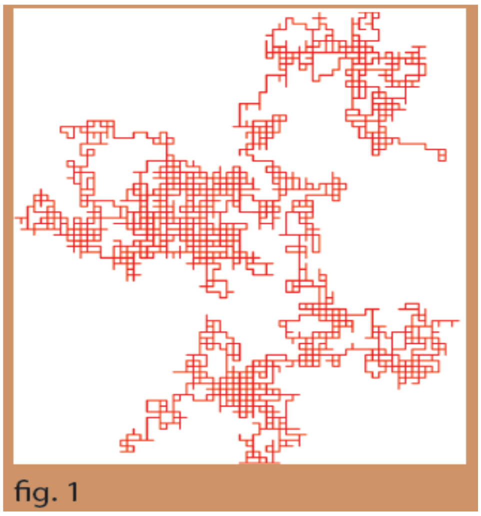


Fig. 1 Random walk in 2 dimensions with 25,000 steps

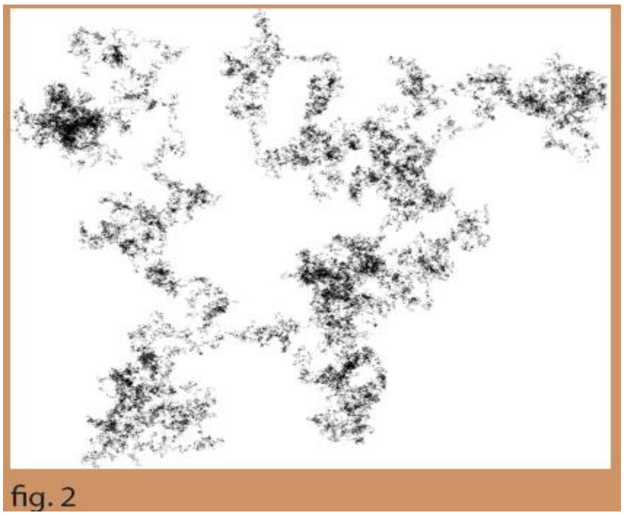


Fig. 2 Random walk with 2 million steps

The random walk we have simulated is in one dimension--you proceed left or right on a straight line--but the model can be extended to two (plane) or three dimensions (space). This type of model can be used to describe the movement of, for example, a butterfly, a gas molecule in a room, or the revenue performance of a stock on the stock market.

Trivia: The term *random walk* was first used in 1905 by the statistician Karl Pearson (1857 - 1936) reminiscent of the staggering, random walk of a drunkard and, as mentioned, is adapted to describe very different phenomena.

THE DIE IS DRAWN (ALEA IACTA EST)

What it is. A die is a polyhedral shaped object used for various games whose marked faces are used to randomly generate numerical or other outcomes. Dice can have different shapes; they can be regular or irregular polyhedra. In a regular die all numbers have the same probability of coming out. The cubic die is the most common, the tetrahedron, shaped like a pyramid, is the least used, the octahedron is the most used after the cube, while the dodecahedron (12 faces) and the icosahedron (20 faces) were used in the past by magicians and sorceresses in their divinatory arts.

Why?

1) Dice are an excellent tool for becoming familiar with the **definition of classical probability** proposed by Laplace (1749-1827).
Assume that all cases are equiprobable. To calculate the probability of an event A:

$$Pr\ (evento\ A)\ =\ casi\ favorevoli\ (al\ verificarsi\ dell'evento\ A)\ /\ casi\ possibili$$

Ex. In a cubic die, the probability of an *even* number coming up is:
favorable cases = 3; possible cases = 6;

$$Pr\ (pari)\ =\ 3/6\ =\ 0.5\ =\ 50\%$$

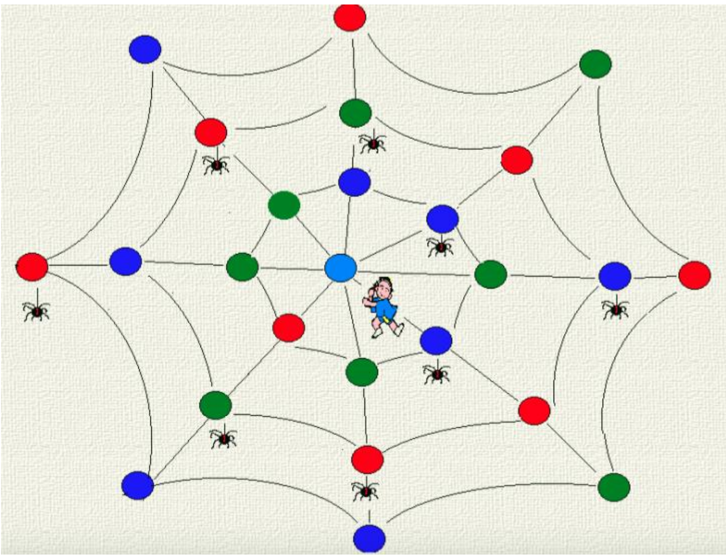
2) But what if the dice were not regular? The outcomes would no longer be equiprobable, and so we could no longer use the previous definition. We could, however, adopt the **frequentist definition of probability,** according to which the probability of an event corresponds to how many times that event occurs out of the total number of trials - made under the same conditions - sufficiently large as the number of trials tends to infinity. We then repeat the roll of an irregular die many times - say

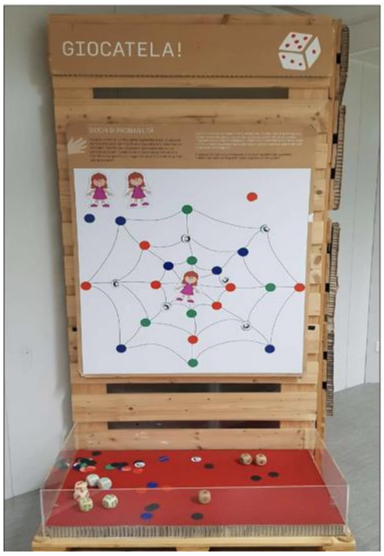
one hundred - and count when times we get as a result a certain number - for example, the number 3. If the number 3 appears 70 times then we can estimate that the probability of outcome three is 70 percent. Our estimate will be more accurate the more repeated trials there are. But be careful, The rolls of the die must all be made under the same conditions.

3) What if I cannot do repeated trials under similar conditions, for example, if I want to assess the probability that it will rain tomorrow? Since we are talking about assessing the probability of a non-repeatable event under the same conditions, we can use the **subjective definition of probability**: the probability assessment of an event depends on the degree of confidence an individual has that that event will occur based on the information he or she has available [De Finetti (1906-1985) and Savage (1917-1971)].

Fun fact: Chevalier De Méré (1607-1684) was a great dice player and it was his passion that led him to consult the greatest mathematicians of his time, such as P. de Fermat and B. Pascal, to submit to them problems related to the game of dice. From these early studies was born the calculus of probability, which deals with random phenomena, i.e., random just like the rolling of dice - "*alea"* is a Latin word meaning precisely the game of dice.

PLAY IT

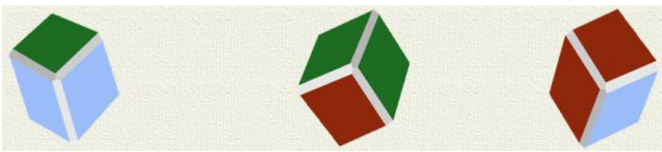




How to play: The little girl imprisoned in the web must find, with the fewest number of steps the way out.

To escape he can move by throwing, at each step, a die of his choice from those available. Each die has colored faces with, in general, two distinct colors present in different combinations. The child can move only on points related to her current position and of the color that came out in the roll of the chosen die. It may happen that some outcomes of the throw do not allow movement.

What we learn: Excellent strategy in choosing the die on each roll.



NUMB3D BY NUMB3RS

DATA

ESTIMATES

How to play the game. The purpose of the game is to guess-that is, to estimate-how many objects there are in transparent Plexiglas containers. Once you choose one of the containers (options are: balls, straws, corks, plastic cups) and make the estimate, you can see exact contents. Also shown is the average and median of the estimates made by visitors to the exhibition so far.
The window that pops up by framing the QR code of this location is as follows:
www.din.usi.ch/pages/stime

INSIGHTS: MEAN, MEDIAN AND MODE

Mean, mode and median are summary indicators that summarize the central tendency of a set of observations.

ARITHMETIC AVERAGE: The arithmetic average is calculated for quantitative variables and is the value obtained by adding numerical data together and dividing the resulting sum by the number of data collected. In common parlance it is used in a wide variety of areas, from the average temperature to the average voter, from the average wage to the average man.

Example: if I have the following observations representing the number of balls held by five boys 6,7,9,12,6 the average is $MEDIA = \frac{6+7+9+12+6}{5} = \frac{40}{5} = 8$ balls. If I wanted to distribute the balls among the boys, giving each the same number and leaving the total number of balls unchanged, I would have to give 8 balls each.
This is an important property of the arithmetic mean: it is that value which, when substituted for the individual observations, leaves the total unchanged.
The arithmetic mean represents the center of gravity of the observations, that is, the sum of the deviations from the mean is zero. In the example above, the deviations of the observations from the mean are:

$$6-8 = -2 \qquad 7-8 = -1 \qquad 9-8 = 1 \qquad 12-8 = 4 \qquad 6-8 = -2$$

Adding up the deviations, we have that $-2 -1 +1 +4 -2 = 0$

OTHER AVERAGES: There are other types of averages, such as the geometric mean and the harmonic mean, which leave other functions of the observations (the product and sum of reciprocals, respectively) unchanged. For further discussion of the **geometric mean** from a historical perspective, see the paper Galileo Galilei and Probability (by Mario Barra) in which Galileo grapples with the following problem:

A horse is really worth 100 scudi: by one is estimated 1000 scudi and by another 10 scudi: it is questioned which of them estimated better and which made some extravagance in estimating.

In the first line Galileo says:
"*I immediately ran to judge the estimate of 1,000 to be more exorbitant, as that to which much greater damage and loss followed*, " but after careful reasoning he concludes:

"*The deviations therefore of estimates from the right are to be judged according to geometrical proportion: and so the one who estimates a stuff the hundredth part of what it is worth, is far more exorbitant estimator than the one who estimates it twice as much or more; and in consequence equally deviate from the right those two who estimate, one twice as much and the other half as much less, one tenfold the right, and the other only a tenth part.* "

Finally, regarding the **harmonic mean**, consider this problem:
The Brambilla family goes on vacation. The average travel speed on the outward journey was 50 km/h. The average speed on the return trip was 25 km/h.
What is the average speed of the trip-that is, that constant speed that I would have to maintain, both on the outward and the return journey, to take the same amount of time?
It is not the arithmetic average of the two speeds, that is, (50+25)/2 = 37.5 km/h but 33.3 km/h.
In fact, if the distance traveled to go on vacation was 100 km, the time taken on the outward journey would be 2 hours while the time taken on the return journey would give 4 hours. A total of 6 hours to travel 200 km.
Then the average speed is obtained by doing total space (200km) divided by total time (6 hours) i.e.: 200/6 = 33.3 km/h. This value is also obtained by doing the harmonic mean (i.e., the reciprocal of the arithmetic mean of the reciprocals) of the two speeds: 1/ [(1/50+1/25)/2] = 33.33 km/h.

MEDIAN: The median is the value that occupies the central place in a data set arranged in ascending or descending order. The median can be calculated for quantitative as well as qualitative sortable variables (such as preference for a drink: don't like it, like it, like it a lot). If the data set consists of an even number of items then there is not one central item, but two. In such cases the median is the arithmetic mean of the two central data elements.
If, for example, we consider the values:
6, 1, 5, 2, 3,3, 4, 3, 4, 2, 0,
corresponding to the number of candies that the 11 students in a certain class eat in a day: the first student eats 6 candies, the second one, and so on.
After sorting the data from smallest to largest, the series looks like this:
0, 1, 2, 2, 3,3, 3, 4, 4, 5, 6;
The median is 3 because the value occupying the middle position, that is, the sixth position is 3. The mean is also 3 in fact the total candies eaten is 33 which divided by 11 (the number of students), equals 3. If each student ate 3 candies, the total number of candies eaten would remain the same.

The median is an indicator of the central tendency of observations that is more robust than the mean with respect to extreme values. In fact, suppose that instead of the value 6, perhaps by mistake we wrote 61, that is, the ordered observations are:

0, 1, 2, 2, 3,3, 3, 4, 4, 5, 61;

The median of the observations remains 3.

Instead, the average increases because it is influenced by the value 61. In this case the total amount of candy eaten becomes 88 which divided by 11 leads to an average of 8.

Another example: if I want to calculate the median age of 7 people I order them from oldest to youngest.

The median age is the age of the person who, in the ordered sequence, occupies the middle position i.e. the fourth position. In the example below, the median age is 25.

60 years old 30 years 28 years **25** years 10 years 8 years 2 years

The play ***the wisdom of the crowd*** can lead to insights related to the fact that the median is more robust than the mean with respect to extreme values. Thus in Galton's experiment with peasants who are well able to estimate the weight of an ox, the mean sums up well the wisdom of the crowd. In the experiment of counting/estimating the number of candies, balls or plastic cups we do in the Let's Give the Numbers! exhibit, it is better to use the median since some children-who do not know well how to estimate the number of candies, balls or cups-tend to give much larger estimates than the actual value.

MODE: The mode or modal value of a data set is that value, if it exists, that occurs most frequently. As the word implies, mode is what most people do. The statement, "It is modeable among boys to

wear jeans," means that "most boys wear jeans." If I look at the length of my classmates' hair and out of 20 students there are 10 with short hair (brush, that is, less than 1 cm), 6 with long hair (below the shoulders) and 4 with medium hair, I conclude that the mode is to wear short hair.
It may happen that there is more than one modal value. Mode is an even more stable measure than mean and median, that is, it does not vary when exceptional values are added. For example, if I have the usual series 3, 1, 5, 3, 1, 4, 12, 7,6, 13, 2, 5, the mode is 3 and it does not change if I add a datum equal to 1000.

SIMULATION MODEL

Background How can we explain the relationship between phenomena that vary in time and space and make predictions about their evolution? One possible way is to build models that can represent reality by introducing abstractions into these models that simplify the complexity of the world around us. This is what statisticians and data scientists do. The Monte Carlo method-named after the city's famous casino-is a tool used to calibrate these models using available data so that the resulting predictions are accurate.

How to play Create an irregular figure by combining the pieces of wood joined by Velcro and place it in the center of the red square that has side equal to 1 m. and therefore area equal to 1 m^2 .
How can you calculate the area of the irregular figure you created?
Here's a possible solution: imagine that you dropped an even rain of ping-pong balls inside the red square: if you counted the balls and found that half of them fell inside your figure constructed from the little sticks, what could you deduce about its area? That it is about half that of the square.
What if the balls dropped inside the constructed figure were one-third of the total balls? You would think that the area of the figure is about one-third of 1 m^2 .
This is a method that gives an estimate of area not an exact value. If I want to increase the accuracy of my estimate I can use smaller objects, for example chickpeas or sand instead of ping-pong balls. Of course, instead of counting chickpeas or grains of sand the can be weighed.
At this point we understand the importance of computers: a suitably programmed computer can simulate the random fall of a large number of chickpeas, can count the number of chickpeas that fell inside the irregular figure and those outside, so as to estimate the area of a generic figure as an appropriate fraction of that of the outer square, which, we recall, is equal to 1 m^2 .
So far we have reasoned with plane figures, but the reasoning can be extended to irregular solids as well. Think of a slice of cheese with holes in it, how can we calculate its volume? Or how can we compare the trunk capacity of a *subcompact* car with that of a *station wagon*? We could calculate the number of suitcases each trunk can hold (the suitcases play the role of ping-pong balls here), and if we wanted to increase the accuracy of our assessment (estimation) we could replace the suitcases with tennis balls ... now don't think about filling the trunk of your parents' car with chickpeas or, even worse, sand, to estimate its capacity - they wouldn't be very happy!

Strategy: You fill the red square (and consequently also the area bounded by the sticks) with a layer of balls. We count the balls (we weigh the chickpeas or sand if we want more precise estimates) inside that red square (including those inside the figure). We denote this value with X. We then count only the balls inside the figure bounded with the little sticks. We denote this value by Y. A simple proportion will lead to an estimate of the area.
For example, let's imagine that in the square we fit exactly 100 little balls. And let's imagine that the little woods delimit exactly ¼ of the area inside the square. Inside the perimeter of the little

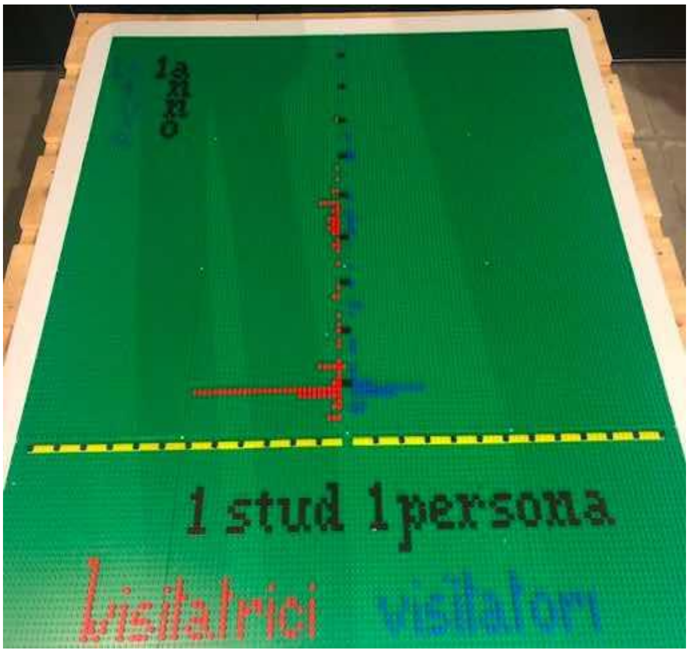
woods will then fit about 25 balls. The corresponding area, denoted by A, is then obtained by solving the proportion:

$$25 : 100 = A : 1m^2$$

so

$$A = 25/100 \times 1\ m^2 = 0.25\ m^2$$

THE PYRAMID OF AGES



At the (NUMBED BY NUMBERS!) "Diamo i numeri!" exhibition, Istat is present with a station where they play with demographic data and compare the age pyramid of the Italian population with that of the exhibition visitors. Let's look at some questions that arise:

- How many boys and girls are there in the most recent generations?
- How many people in Italy are over 90 years old?
- Are more girls or boys being born?
- Are there more elderly or elderly women?
- What is the largest age group?
- And are the visitors to the exhibition the same as or different from the Italian population in terms of age and gender?
- Among the visitors to the exhibition can we expect more boys than girls?
- And who accompanies them most frequently? Adult people such as mom and dad or teachers, or young kids such as older siblings, or even older grandparents?

Some of these questions can be sought to be answered in the pyramids of ages.

The post consists of:

- Pyramid of the ages of Italy in 2018 built with lego bricks
- Pyramid of exhibition visitors' ages built with lego bricks

What is the Pyramid of Ages

The Age Pyramid is a graphical representation of the age and gender distribution of a population. It consists of two bar graphs rotated so that they share the same vertical base in the center of the graph: the one on the left represents the age distribution of the female population and the one on the right the age distribution of the male population. Along the vertical axis are the ages in completed years (1 button = 1 year) and on the horizontal axis are the absolute frequencies of men and women (1 button = 10 thousand people in the Italy pyramid, 1 button = 1 person in the visitors pyramid), corresponding to each age considered.

From the shape of the pyramid, it is possible to derive indications of both the factors that characterize the age and gender structure of the current population and past trends. Forecasts for a time span of no more than a century are also possible.

Such indications can be drawn by analyzing the following elements of the pyramid:

- **the base**: gives indications about the flow of births: if it widens, there is a sharply increasing flow of births; if it narrows, it means the flow of births is decreasing.
- **the slope of the sides**: gives indications about the general level of elimination by death: if the slope of the sides is strong, there is high mortality; if it is weak, there is low mortality.
- **the presence of bulges or bottlenecks for particular age groups**: provides an indication of the intervention of particular disruptive factors, for example as happened during the world wars.

THE AGE PYRAMID OF ITALY IN 2018

The age and gender structure of Italy's population in 2018 has lost the characteristic pyramidal shape, typical of high-birth, high-mortality populations, like Italy before the demographic transition and like many developing countries still today. The distribution of the Italian population in 2018 presents a characteristic "spinning top" shape, with the lower part, corresponding to the young population, thinner, a heavy central body corresponding to the adult population born around the 1960s, and a second refinement in the upper part corresponding to the elderly population, where a gender asymmetry in favor of the female component, which is longer-lived, is particularly evident. It is therefore an aging population, characterized by low and declining birth rates and low mortality, which increases survival even at advanced ages.

THE AGE PYRAMID OF VISITORS TO THE EXHIBITION

Each visitor inserts his or her own button into the Visitors' Age Pyramid, placing it corresponding to his or her age in completed years, on the left (the red part) girls, girls and women, and on the right (the blue part) babies, boys and men.

This will give the age and gender distribution of visitors. Did more men or more women visit the exhibition? More young people, adults or the elderly?

The tablet contains 2 navigable sites, 2 videos and 1 document:

1. **United Nations World Population Prospects**: graphical representations of the population profiles of all the countries of the World. (https://population.un.org/wpp/Graphs/DemographicProfiles/)
2. **The Pyramids of World Population and All Countries of the World from 1950 to 2100** (https://www.populationpyramid.net/it/mondo/2060/)
3. **The Age Pyramid of the Evolving Italian Population from 1972 to 2061** (http://www.istat.it/it/files/2011/05/piramide.wmv)
4. **The Age Pyramid of the Italian population from 2011 to 2065 with the component of resident foreigners and working-age population highlighted**
5. The **press** release / The **country's demographic future to 2065.** Regional forecasts of resident population to 2065. (https://www.istat.it/it/files//2018/05/previsioni_demografiche.pdf)

CAPTURE-RECAPTURE

The historical origins of the capture and recapture method

The first to use the method was, in 1802, Pierre Simon Laplace to estimate the population of France, based on his estimate of those born in France in a certain year and those born and residing in certain French communities that were particularly orderly and accurate in administrative management.

For a second documented application we have to wait almost a century, 1896, when Danish marine biologist Johannes Petersen, estimated the numerical presence, in a given stretch of sea, of flounder.

The population estimation formula by the capture and recapture method is also known as the "Lincoln-Petersen estimate" or "Chapman estimate."

How does the capture and recapture method work?

The "capture-recapture statistical method" in English ***capture-recapture*** or also ***capture-mark-recapture***, is a statistical method used in the biological sciences to estimate the size or abundance of an animal population or other biological group in a natural environment. This method is often used to estimate the number of individuals in a species when it is difficult to count them directly. It is widely used in ecology and conservation biology.

The capture and recapture methodology is based on a process in which animals are captured, tagged (e.g., with a non-harmful marking such as a sign or bracelet) and then released back into their natural environment. Next, some animals are captured again and the number of tagged individuals among those captured is recorded. This information is then used to estimate the total population.

The method is also used in Medicine, to estimate how many individuals are affected by a certain disease, such as diabetes.

The statistical basis of the method

Let us try to retrace the estimate made by Petersen in 1896.

- In the stretch of sea we are interested in, there will be a number of **N** flounders, not known in advance.
- We walk the stretch in question, fishing in the net **n** flounders. We mark them in some indelible way and throw them back into the sea.

- Let's give the flounders caught the first round a few days to recover from the scare and get back to everyday life, and carry out a second round of fishing. This time we will catch **K** flounders
- Among them, we will find some, **k**, marked on the first round.

To reason about the numbers we have collected, we need to assume some ideal conditions:

- first hypothesis: that the flounder population in that stretch is **closed**, i.e., that it remained unchanged in the days between the first and second capture, thus with no births, no deaths, and no *immigrants* and *emigrants*;
- second assumption: that individuals have an equal probability of being captured and then recaptured; thus, the capture criterion should not favor particular characteristics of individuals in any way;
- third hypothesis: that marking does not alter the behavior of marked individuals; thus, for example, that the lived experience does not make them more guarded, prompting them to avoid subsequent recapturing.

But then, how many flounders are there?

At this point it can be assumed that the recaptured sample is representative of the entire population. Then the fraction of tagged flounders relative to the entire flounder population ($\boldsymbol{n}$ / $\boldsymbol{N}$) can be assumed to be equal to the fraction of recaptured flounders that are marked ($\boldsymbol{k}$ / $\boldsymbol{K}$). It is then sufficient to solve the equation in the unknown **N**:

$$N = \frac{n \cdot K}{k}$$

Example. Suppose 100 flounder (**n = 100) are** caught in the first round, which are marked and gently thrown back into the sea. In the second round, 100 flounders (**K = 100) are** caught again, of which 5 are found to be marked (**k = 5**).

The estimate of the number **N of** flounder in that stretch of sea is therefore:

$$N = \frac{100 \; x \; 100}{5} = 2.000$$

This is an estimate and not an exact value, evidently, because a statistical sampling method is used. How likely the estimate is depends both on how realistic the three assumptions made are and on the number of individuals examined. There are refinements of the method to improve the likelihood of the result, even in the presence, for example, of a non-closed population, but the method still returns an estimate.

In the post at the exhibition

How to play: The goal is to estimate the number of white ping-pong balls in a container with white and yellow balls. Successive handfuls can be taken. A pencil is provided.

Strategy: In the first handful mark all the selected white balls with an x in pencil. You then put them back into the container and shuffle the balls. In the second handful you count how many balls marked with x were again selected. Again a proportion is used to estimate the number of white balls.

Other external resources:
https://olmo.deib.polimi.it/ecologia/dispensa/node30.html

COUNTERS

From http://www.worldometers.info we bought the 4 meters the following meters:

1) World population
2) World public spending (in USD) on education today
   ("today" starts at midnight on the clock of the computer on which the counter is installed)
3) Hectares of forest destroyed since the beginning of the year
4) Days to the end of oil reserves

On this site you can find various other meters and can make interesting comparisons.
For example, education spending in the world is less than health spending and about twice as much as military spending.
Here is some information on education spending in Italy
https://www.ilsole24ore.com/art/notizie/2017-08-29/italia-terzultima-europa-spesa-istruzione-germania-spende-doppio-190050.shtml?uuid=AE8jEVJC

MACHINES THAT ANALYZE IMAGES

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Artificial Intelligence systems capable of analyzing images by emulating typical human vision behaviors are becoming increasingly popular nowadays. Such systems are able to support a large variety of applications such as *face analysis and recognition, robot guidance systems, video surveillance, area inspections using Unmanned Aerial Vehicles (UAVs), etc*. The exhibit shows an Artificial Intelligence system that by analyzing a video detects the presence of a face and through a series of control operations on its knowledge base is able to associate the detected face with an identity that will be displayed on the screen. In case the system has no information about the examined face it will label the individual as *unknown*. Attention is drawn to how all that is needed is *a processor, a camera* and knowledge of a programming language such as *Python* and the *face recognition* library[1] to implement one's own facial recognition system. Systems capable of autonomously recognizing a face represent a technology, studied as far back as 1965, but continually refined over the years to the point of becoming commonly used. The low error rate of such technologies has allowed their use in a variety of contexts acting on the change of our way of life, in fact, commonly we can observe the use of the face as a key to unlock the *smartphone* or to access our personal banking area, we have the ability to search the entire Internet for a face whose information we want to receive, and many Social Networks are able to alert us when we appear on a photo uploaded by another user. Such technological advancement brings with it a number of positives and negatives that the visitor is invited to reflect on. There are an estimated 245 million surveillance cameras installed worldwide, and with the accuracy of today's facial recognition systems a machine could, for example, easily track the movements of any individual potentially attacking his or her *privacy*. Software exists that can recognize one person's face among that of thousands in less than a second, and this recognition is often done in the total unawareness of individuals.

In more detail, a Face Recognition System is a technology capable of identifying a person from an image or video. The system analyzes images that are captured by the image/video capture system pixel by pixel by going looking for shapes that suggest the presence of a face. Its operation is based on Algorithms that extract features of the face detected in the input image and then compares them with those extracted from the face present within another image. Some of these features could be, for example, the *position, size and shape of the eyes, the distance between the nose and mouth, and the size of the face*. The extracted information will be processed through the use of mathematical functions that will allow the system to figure out how much two sets of features differ by being able to recognize a known face. If the differences found are negligible, it can be reasonably assumed that the two faces belong to the same person.

VIDEO PROJECTION ROOM

There are 4 videos in this room:

- one produced by CSCS - Swiss Center for Scientific Computing
- two products from ISTAT
- one produced by Prof. Antonietta Mira in collaboration with Paolo Zanocco and Ilaria Curti entitled Angels of Big Data.

Angels of Big Data illustrates the opportunities and fears that big data (large masses of data that we ourselves generate as we entrust our lives to technology) generate.

A lecture by Prof. Antonietta Mira on the topics illustrated in the video can be found at this link.
https://www.youtube.com/watch?v=kItlFRDF5XU
https://www.brainforum.it/video/big-data-lombra-inquietante-del-grande-fratello/
https://www.youtube.com/watch?v=T1eH2yhjHZc

The two ISTAT videos can be viewed at these links:
the first is for adults: https://youtu.be/OSGs1FVhkpY.
The second is for the boys: https://youtu.be/awdHmoWyFuE.

This is the website of the Swiss Center for Scientific Computing (CSCS), which produced one of the videos of the cinema hall:
https://www.cscs.ch/about/about-cscs/

NUMBED BY NUMB3RS

STORIES BY NUMBERS

HISTORY OF URI

This is a story that happened a long time ago in a village in Mesopotamia, where lived the Sumerian people of whom little Uri was a part. Among the noises and sounds coming from the village, a call could be distinguished now and then: - Uri! Uri! It was Mother who had some errands for him: - I need some cottage cheese. - Yes mother. How many? - In those days men still couldn't read or write. They could count a little, helping themselves with their fingers, not as much though and not as well as us. Mother then showed with her fingers how many cottage cheese she wanted. With his little hand Uri would carefully copy Mama's gesture and ... off in a hurry! And with the basket full Uri would return home to deposit the cottage cheese and then run off to play until the next call.
- Uri! Uri! - Yes mom? - Some eggs. - How many? Uri reached out to arrange her little fingers one by one as Mother showed and then- off she ran! As Uri was running along the path, here comes a beautiful butterfly passing by and with two nimble leaps Uri catches it; lifting his arms to the sky Uri pauses for a moment to watch the butterfly as it moves away from his hands and ... - O nooo! The hands! Yeah: and now what? Uri had to go back to Mother to see again how many eggs were needed. Mother then showed her fingers again, and Uri again departed. But either for a colorful butterfly or a beautiful shiny stone, the times Uri could get where he was going without losing count were very few. Here he needed an idea. One afternoon that Uri was at his usual stirring with his hands in the clay of the shore, the idea came. - Why, sure, just make my fingers out of clay and I'll use them to count and finally have my hands free to play! Now when Mama showed him how many measures of barley were needed, he would put a cone next to each of Mama's outstretched fingers, collect them all inside the bubble and ... off in a hurry!
Once, perhaps it was because the time of the big village festival was approaching, Mother asked Uri to bring her fingers and fingers and fingers and more fingers of ripe dates. All these fingers were beginning to be cumbersome, and Uri could no longer hop quietly here and there.
Thinking and rethinking, one hot afternoon as Uri stood on the riverbank kneading new fingers the idea came: instead of all these fingers I will make balls. So he reshuffled the clay, forming many nice balls not too big so that they would not weigh. From then on whenever in his count fingers Uri could put two full hands together he would make them disappear and put a ball in its place. Uri would now leave with his little pile of three or four little heads plus a few loose fingers.
But there was no end to the difficulties: what about when he would get the sudden and compelling urge to do a somersault? If he did not restrain himself it would be trouble again: cones and balls scattered everywhere.
Thinking and rethinking, another idea came. - Of course! I will make big balls out of soft clay, crushed then like scones. There I will make the footprints of the pebbles that I will leave at home in the bubble. Just the footprints! So Uri between jumps and somersaults could throw in the air and pick up the clay diaper without any more worries.

The problems, however, were never ending. If Uri was now older even mother's errands were more difficult. And so she needed nuts, figs, bowls of milk and sacks of flour, right away, all at once, and

of course each thing had its own number of fingers. How to do it? Carry a loaf of bread for each? Goodbye free hands! And then how to remember which loaf was for nuts rather than milk or figs or flour? An idea was needed here. Thinking and rethinking one evening at sunset, as Uri sat along the riverbank, scratching the clay with small twigs, the idea came. - But sure I will draw on my loaf the nuts, the figs, the milk, the flour and ... next to the pebbles. Choosing a nice straight stick Uri wrote, hear well for the first time he wrote, what mother had asked him. This had been a really but really great idea. From that day until today things have changed a little bit. Today we have letters, numbers, paper and pens. But what if Uri's idea had not come that night? Who knows.
Meanwhile, with his buns, Uri, who had now grown up, was doing very well on all errands.
Now not only Mother, but everyone, even the king's aides and the king himself, turned to Uri when there was something to count.

Excerpted from: *Uri, the Little Sumerian* by Raffaella Petti , 2009, The Garden of Archimedes.

THE TRIUMPH OF ZERO.

Once upon a time there was
a poor Zero
round as an o,
so good but yet
he counted really zero
and no one wanted him in company
so he wouldn't throw himself away.
Once by chance
he found Number One
in a bad mood because
he could not count
to three.
Seeing him so black
little Zero plucked up
courage,
in his car
offered him a ride,
and stepped on the accelerator,
very proud of the honor of
having
such a character
on board. Suddenly
who can be seen
standing still on the sidewalk?
Mr. Three taking off his hat
and
bowing-and then, by Jove,
Seven, Eight, Nine
doing the same.
But what had happened?
That the One and the Zero
sitting side by side,
one here the other there
formed a great Ten:
no less, an authority!
From that day on, the Zero
was highly respected,
indeed by all numbers
sought after and courted: they
zealously and solicitously
yielded him the right,
(of keeping him on the left
they were afraid),
invited
him to
dinner,
paid for his cinema,

for little Zero
was happiness.

Excepted from: *Il trionfo dello zero* by Gianni Rodari, 2011, Emme Edizioni.

THE PROBLEM OF FOUR

One day the number four got tired of being even. Odd numbers, he thought, are much more cheerful and witty. And he got tired of its somewhat bland, seat-like shape. And he got tired of being two plus two, which everyone knows, and indeed when everyone wants to say something everyone knows they say, How much is two plus two?
Look at seven, he told himself, so quick and elegant, and three so round and witty. He dreamed of being an odd number, like three, like five, like seven.

Such a problem the four did not know how to solve. Perhaps it did not even have a solution. If he had it though, the Great Mathematician must have known it. So four went to the Great Mathematician and presented his case to him. The Great Mathematician smiled. He too had once wished to be different: not another one, of course, because he wanted to remain himself, but a little more like the Great Musician, or the Great Director, or the Great Tennis Player. He, too, therefore, had had the problem of four and knew how to deal with it.

She made him sit on the floor (a chair would really be useless) and began to talk to him. - You see, four," he said, "there is no need to become different, to become odd for example, or long and difficult. There is no need because you are already different and unique even if you don't realize it. To you it seems that you are a stupid little seat that puts two and two together and everyone knows it, and instead there are things in you that no one else has, very special things. For example, you are two plus two, but also two for two and also two to the second. And that is a totally extraordinary fact. Three plus three is not also three for three, much less three to the third! And then you who so admire three, do not realize that it is already within you, because it is true that you are two plus two, but it is also true that you are three plus one, the first odd numbers of the whole infinite number chain. -

At that point number four was feeling quite confused and even a little upset, because that business of already having in himself what he was looking for outside seemed to him to be something really important and something to think about. He begged the Great Mathematician to stop and left. Since then, number four has realized that he matters much more than he thought, and every day he discovers that he is always different and likes himself that way.

Taken from: *Philosophy in fifty-two fables* by Ermanno Bencivenga, 2011, Mondadori.

"There is no such thing as luck: there is the moment when talent meets opportunity."

Seneca (4 B.C. - 65 A.D.)

STATISTICS OR LUCK?

A WAY TO WIN BY PLAYING DICE

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The paper describes the experience of an activity provided by the National Plan for Scientific Degrees (PLS) carried out at the Benedetto Croce Scientific High School in Palermo. It is an activity with the dual purpose of fostering knowledge of Statistics to young students and orienting them toward a more informed choice of university education.

The Perudo: From Game to Statistics. When the dice don't count

Perudo is a dice-based board game originating in Peru - hence the name - dating back to the time of the Incas. In truth, it boasts varied origins, including among many North American Indian peoples. From Peru it would come to Spain, then spread to the rest of the world in the 15th century through the Conquistadors. It owes its celebrity to the movie "Pirates of the Caribbean - The Curse of the Phantom Chest on the Flying Dutchman Ship." Some of the protagonists such as Bootstrap Bill, Davy Jones, Will Turner to decide the latter's stay on the mysterious ship entrust the decision to the winner of the game deemed, just by virtue of the fact that he or she turned out to be a winner at the game, skilled in reasoning and deduction.

It is described here as an example to recover the usefulness of statistical thinking (which allows-luck permitting-to win at the game) that emerges almost unconsciously among players. The example is given because in the classroom one can proceed with the game among students and follow a *brain storming* approach to verbalize and focus all the statistical concepts and information possessed or to be conveyed.

The object of the game is to remain the last player in the game with at least one die. There can be from 2 to 6 players. Each player is provided with a compass with 5 dice. Faces 2 to 6 are depicted on the dice. The number one face is replaced by the Inca or llama face.

It starts with a compass with five dice each. The dice are all shuffled together and each player looks at his or her own dice without showing them to the opponents. The actual game consists of a series of raise bets on the total number of dice (occurrences) with the same value (the same face of the dice) among all players.

In turn, the first player, after seeing the result of his or her dice, declares a bet on a value of the dice considered most frequent and on the occurrences of that value. The hand player must first decide whether he wants to raise with a different bet or wants to evaluate the previous player's bet. Declarations/bets are based on the evidence of one's own dice and the declarations of previous players. Following these declarations, it is possible to assume the presence in the table of a certain number of dice with a certain value.

Before making the bet, it is necessary that you keep in mind:

a) How many dice in total are in play. At each turn, as the game goes on, dice are lost and thus eliminated from the game. By the time the losing players also lose dice, more experienced and careful players will be able to remember exactly how many are left in play, and thus make their bets/hypotheses with greater precision and accuracy;
b) all Inca or llama heads count as jokers. This makes it even more difficult to make likely and therefore winning guesses.

The next player may raise the bet with a higher occurrence on the same die value or with the same occurrence but a different die value than the previous player. For example, if the first player at the beginning of the game declares:
10-5 is indicating that based on his dice (the only ones to be known by him because they are displayed) he generalizes that among all players there are 10 dice with face #5.

In the first case, he can make his assumption starting from 10-5 and then he can:

a) Bet on 10-x where x is one of the dice faces other than 5;
b) Betting on a number greater than 10-5 (e.g., 11-5),
c) finally can halve the number 10 and bet 5-lama, that is, the wild card.

In the second case, the hand player may evaluate of the bet made by the player before him. In that case he is said *Dubito* if the summation is considered false; or he may consider the bet to be correct: in which case he will say *Calza*.

In general to say *Stocking* means to state that the previous statement is correct. If the statement is correct the player who socks acquires a die and the one being challenged loses it, otherwise it is the player who socked who loses it;
To say *Doubtful* is to say that the previous statement is not true because the number of actual dice is believed to be less than the number of declared dice.

If the statement is correct the player being challenged loses a die, otherwise it is the player who doubted who loses it.

The moment a player has said Dubito (in the case of a bet deemed too risky) or Calza (in the case of a bet deemed congruous), all players are required to uncover their dice and the veracity of the last statement/bet is verified. One can bet on Inca or llama heads at any time during the game by dividing the last occurrence bet by two.

If the hand player has preferred to make a guess, the game passes to the next player who is in the same condition as the previous player.

When a player remains in the game for the first time with only one die there is declared to be a *palifico*. In this case the palifico will have the right at the beginning of the round to choose the value to bet on for the entire round, which, therefore cannot be changed. During the round, blade heads do not count as jokers. It is possible to declare oneself a *palifico* only once in the entire game.

Each round ends with either the loss of the die by the player who misdeclared or misdoubted or the winning of the die by the player who identified the exact number of occurrences and value by saying sock. In the former case, the next round begins with the player who lost the die, otherwise it proceeds clockwise. The one who has socked begins if his or her bid was successful.

Below is the formalization of the steps and decisions to be taken.

$$\mathrm{d} = \mathrm{dichiarazione}$$

$$\mathrm{F_i} = \mathrm{dadi\ sul\ tavolo}$$

$$\mathrm{f} = \mathrm{dubito}$$

$$\mathrm{v} = \mathrm{calza}$$

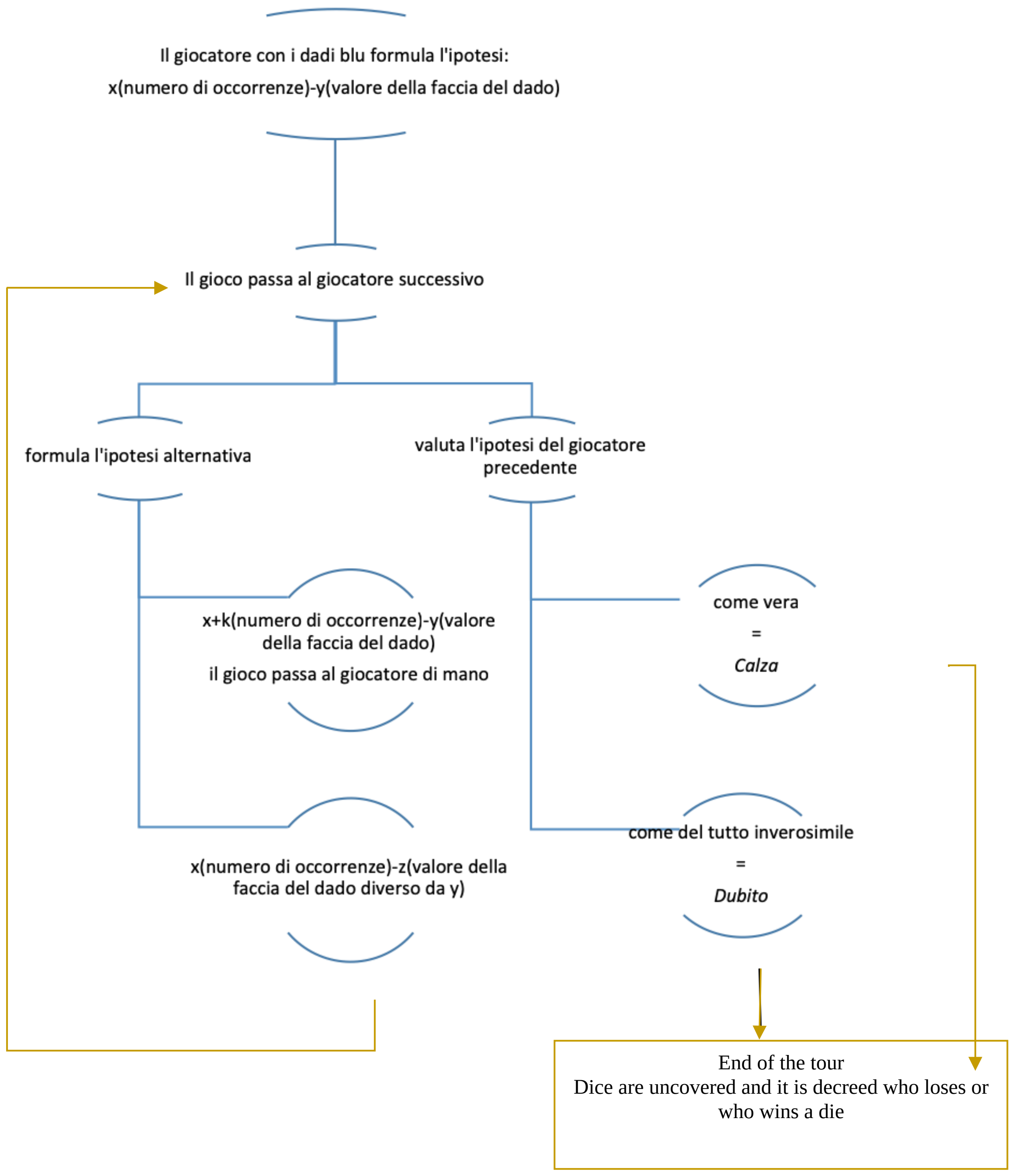
$$\mathrm{n} = \mathrm{n^\circ\ di\ dadi} = 30$$

$$\mathrm{F_i} \rightarrow \mathrm{i} = \mathrm{n},\ (\mathrm{n}-1), (\mathrm{n}-2), \ldots,\ 1.$$

$$\mathrm{f}\begin{cases} \mathrm{vince} \rightarrow \mathrm{d} < \mathrm{F_i} \\ \mathrm{perde} \rightarrow \mathrm{d} \geq \mathrm{F_i} \end{cases}$$

$$\mathrm{v}\begin{cases} \mathrm{vince} \rightarrow \mathrm{d} = \mathrm{F_i} \\ \mathrm{perde} \rightarrow \mathrm{d} < \mathrm{F_i} \vee \mathrm{d} > \mathrm{F_i} \end{cases}$$

Outline of the procedure of the game and the decisions to be made:



Therefore, the number of dice wagered on is counted. The player who declares stocking wins if the number wagered as stocking is exactly the same as the number in the table. In this case, the win consists of the recovery of any dice previously lost. If the same player, on the other hand, declares Doubt wins if it turns out that the doubted number recurs a different number of times than the previous player bet. In this case, he will win the bet and it is the previous player who will lose a die.

In other cases, the person who pronounced the bet wins. In counting face occurrences, the inca or blade heads, which count as wild cards, take on the value of the die whose occurrence is being counted. The last person left with at least one die wins.

Perudo is a dice game and therefore seems to have more bearing on Probability instead of Statistics. Although an important role must be attributed to luck in dice, one does not win with luck alone! A certain strategy must be established to choose the steps to be taken. Most experienced Perudo players rarely doubt the bets of others at the beginning of the game, but rather wait until the bets are so high that they risk being challenged themselves by another player. It is appropriate to bet on the Inca or llama heads when the bet is not high enough to require the *Dubito*, but is too high for it to be increased with reasonable confidence. The art to winning in this ancient Peruvian game lies in knowing whether and when to make a personal bet or to evaluate that of the player before him.

And here is where Statistics comes in. Translating Perudo into concepts of statistical inference, we can identify:

a) the value of the die and the correct occurrences to be found to win is the estimator;
b) the actual value of the die with exact occurrences is the unknown parameter;
c) displaying one's dice and thus one's number of occurrences and value of the most frequent die is the sample estimate that gives rise to a sample distribution;
d) the bet is the null hypothesis that needs to be assayed;
e) the next player's bet is the alternative hypothesis;
f) the risk of losing by saying "*I doubt*" is α the error of the first kind (rejecting the null hypothesis when it is true);
g) the risk of losing by saying "*Stocking*" is β the error of the second kind (accepting the null hypothesis when it is false);
h) the probability of winning by saying "*I doubt*" is 1-α the significance level of the hypothesis;
i) the probability of winning by saying "Stocking" is 1- β the power of the statistical test;
j) the most likely bet is the expected value of the faces (obtained from the sum, being independent events, of the expected value of the individual faces and the expected value of the blade faces). And the bet/hypothesis is likely if it is supported by the evidence. As the dice are lost the expected value changes because the number of possible occurrences and probable occurrences change;
k) the hypotheses/betters of others can give strong indications for inferring the most likely hypothesis. The expected value is just a summary of the frequency distributions of everyone's statements, which, if they are not bluffing, first try to approximate the modal value of occurrences and then the expected value (as the number of statements increases, the modal value tends to the expected value);
l) the number of llamas (Jolly) in the game could be identified as an outlier value to be taken into account when making the null or alternative assumptions.

The formulation of hypotheses from only partial information regarding the dice view in one's possession can provide the cue to introduce the concepts of partial information or detection and census information or detection in the classroom. In addition, bets can gradually be refined as one listens to each player's statements, making each bet strongly linked and conditioned to the previous bet. Conditional probability or set theorems can safely take their cues from the last considerations.

AS AHMES, ASHA AND MAYA BECAME FRIENDS AND LEARNED TO "COUNT" ON EACH OTHER.

(by Antoinette Mira)

3700 years ago an Egyptian boy, Ahmes-who at age 40 became a famous scribe and transcribed the Rhind papyrus (1650 B.C.E.-and a young Indian girl, Asha, attended the same school.
Impossible, you will say. This is a fairy tale where anything is possible! Just close your eyes and let your imagination fly.

On the first day of school Asha arrived in class with a bag full of mangoes.

"If we want to become friends, we have to learn to understand each other"-Asha told Ahmes-"let's start with a common system of symbols to indicate numbers, so every day I will bring you fruits and we can count how many I have given you."

"You gave me a mango today," Ahmes said in thanks, and drew a vertical line in the sand

I

"Even the Brahmic script of ancient India in the 3rd century B.C. used the symbol I to denote the number 1!" exclaims Asha, proud of her ancestors.

"How old are you Ahmes?"
"Ask the reader!" he replied, winking ... at the reader of course.

Number 1 is thus written similarly by both Ahmes and Asha: a solid starting point for their friendship. This is not surprising. Virtually all cultures today, as well as in the past, use a similar symbol to represent the 1 just as we do. The practice is tens of thousands of years old and long predates writing, which is only five thousand years old. It seems to have been started by hunter-gatherers who, instead of writing in the sand as Ahmes had done, used bones or sticks to record quantities by carving indentations so that a trace would remain even after a rain that would erase the marks drawn on the ground. Carved bones and sticks also had the advantage of being able to be transported from place to place, and you know, hunter-gatherers were nomadic: when a land was no longer fertile or animals dwindled due to migration, they too moved from place to place in search of new food.
"Let us now turn to number 2," Ahmes continues and traces two lines on the ground with his finger to indicate that, on the second day of school, he received 2 juicy mangoes as a gift from Asha:

II

And the next day, to write number 3 Ahmes uses, of course, 3 lines:

III

Unlike their contemporaries, such as the Mesopotamians, the Egyptians did not group their I's in specific patterns. So for example, the three bars of Ahmes can be placed either horizontally or vertically.

So far, so simple. But think about the number 7: It was easy to get confused when reading it, especially from a distance:

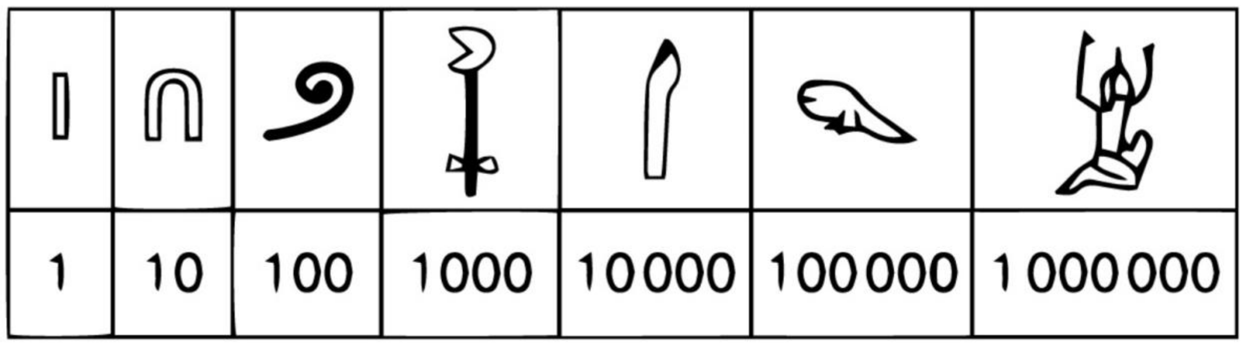
IIIIIII

Asha then suggests some shortcuts: "If we take the two bars representing two mangoes and put them horizontally, so much for you it makes no difference, and as we trace them in the sand, to be faster in writing, we just raise our finger, that's what happens."

Thus was born the symbol representing the number 2 as we know it today.

And in a similar way we have for example the number 3 and 7 more or less as we would write it today:

While it is easy to carve a straight line with a knife on a piece of wood, carving a curved shape, like our 2, would be unnecessarily difficult. So Ahmes found Asha's idea interesting, but stuck to his symbols.
Ahmes agreed, however, that a new symbol was needed for the number 10 so that it would not be confusing to count all those sticks one in a row. So too for the number 100, 1,000, 10,000 and 100,000. A special symbol was then needed to indicate very large numbers ... and we will come back to that.



Symbols used by Ahmes.

Ahmes had an interesting way of indicating addition and subtraction operations through a symbol in hieroglyphic writing. It represented a man running: toward the quantities if they were to be added, in the opposite direction if they were to be subtracted.

Ahmes also had a special symbol, which we would call "zero" today. But zero was used by Ahmes only in architecture to indicate the ground floor of a building such as a pyramid. To count the floors below ground Ahmes used what we today call "negative numbers." Now, as then, it was enough to put a particular sign in front of the number to indicate that that floor was below ground. Second floor below ground: -1 we would write.

Asha found the idea of the "zero" symbol very interesting and decided to make even wider use of it, not only in architecture but in general to represent numbers in a totally different way, the positional number system.

This idea was secretly passed from Asha to Brahmagupta a mathematician and astronomer who in the 6th century used a small black dot to denote this new symbol. Thus was born what we call zero today.

To indicate the number one million Ahmes depicted a man with his arms raised toward the sky. In general this symbol was used to represent extremely large numbers as we would do today if we wanted to indicate with our arms that we love someone very much: we would spread them wide to symbolically contain all our love!

Maya, who attends the same school and comes from Guatemala-in Central America, the cradle of Maya civilization as early as 750 B.C. - wants to have his say on systems for representing numbers, on the strength of the fact that the Maya had devised a sophisticated counting system to take into account, for example, the time between two religious rituals, between particular astronomical events such as eclipses, between periods of greater or lesser fertility of the earth, or again, to avoid embarking on journeys on unlucky days and instead arrive at their destination on auspicious days.

"We have 10 fingers and 10 toes, so why not count in base 20?" asked Maya. It actually seems very natural, considering also that in those ancient civilizations people mostly walked barefoot and therefore toes ... were within reach! "Explain your idea better," asked Ahmes and Asha.

"Let's imagine that we only have three symbols," Maya continued:
. points, FRIJOL (bean), for the unit
_ lines, PALITO (stick or toothpick) for #5
O and a shell, the symbol for zero

"In my opinion, only three symbols are too few to write all the numbers!" argues Ahmes who was used to using many more.

But here is Maya's brilliant idea.
The symbol changes value depending on its position. An idea the Egyptians had not thought of, and, in fact, for Ahmes the order in which the symbols were written was irrelevant.

Imagine, Maya said, building a temple of numbers-and you know, the Maya were masters of building magnificent temples.

As one ascends to the higher floors in a temple one gains value, and indeed, at the top of temples we usually find precious altars for sacrifices to deities.
Now we think that the symbols on the higher floors-written above-are worth more than those on the lower floors. Each time you go up a floor the symbols are worth 20 (like fingers and toes combined) times more.

For example, a bean on the ground floor is worth 1:
. one

A bean on the second floor is worth 20 times as much, so it is worth 20.
Here then is the number 21:
. twenty.
. one

And the number 22
. twenty.
... two.

And again number 41:
.. twenty x 2 = forty
. one

And, if we go up another floor here is number 421

. 1 x 20 x 20 = 400 = four hundred
. twenty.
. one

The same applies to the symbol "little wood": if it is placed on the ground floor we have the number 5:

_ five

But if the little wood goes up one floor here is where its value goes from 5 to 5x20 or 100. And so Maya writes the number 105:
_ 5 x 20 = 100
_ 5

And the number 101 becomes:
_ hundred
. one

And here is the surprising and magical usefulness of the zero: to write the number 20 you need a special symbol, the zero in fact!

The number 20 is then written by Maya as follows:
. twenty.
0

Number 400:
. four hundred
0
0

And if we add a bean on the ground floor we reach 401:
. four hundred
0
. one

If we put a stick on the ground floor instead, we have the number 405:
. four hundred
0
_ five

Easy exclaimed Asha! Who decided to keep Maya's "positional" idea (symbols have a different value depending on position), but wanted to change the base from 20 to 10 (thus using only the fingers of the hands as a reference) and instead of just 3 symbols decided to work with 10 symbols.

Instead of ascending to higher planes, Asha symbols change value by moving to the left, and each time a symbol moves left one position its value is multiplied by 10

1
10
100

And the same for symbol 5:

5
50
500

"Easy isn't it?" cried Asha happy with her new and elegant way of representing numbers.
And so it was that Ahmes, Asha and Maya became best friends, lived happily ever after being able to rely on each other...for eternity, that is, for an infinite time.
And about infinity we will talk in another story!